Identity Criteria for Structures

Christopher von Bülow*

August 6, 2005

The question of what structures-as-universals are can be approached in a further way, namely, by searching for an identity criterion for structures. There is a restricted and an unrestricted version of this goal: in the restricted version, we look for a criterion which determines for arbitrary structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$, say, the respective structures of given systems S_1 and S_2, whether they are identical; in the unrestricted version, we want a criterion which determines identity for arbitrary structures and arbitrary *entities*, say, Julius Cæsar. Unrestricted identity criteria for structures will not be addressed here at all. For restricted identity criteria there are different candidates, the most obvious of which is isomorphism. The corresponding candidate identity criterion would be

identity criteria for structures
restricted …
… and unrestricted
restricted identity criteria: isomorphism?

(I_0) S_1 and S_2 have the same structure, i.e., $\mathfrak{S}_1$ is identical with $\mathfrak{S}_2$, iff S_1 and S_2 are isomorphic.

But although isomorphism is certainly a sufficient condition for having the same structure, it is very plausible that it is not a necessary condition. This is easily seen by looking at the example of lattices. This type of structure can be characterized in two different ways.[1] On the order-theoretic characterization, a lattice is a non-empty set A of objects together with a (partial) ordering[2] '$\leq$' on A such that for all a, $b \in A$ there are infima[3] as well as suprema[4] of $\{a, b\}$. The other way of characterizing lattices is algebraic: here, a lattice is a non-empty set A together with two binary operations $\cap$ and $\cup$ on A which are associative and commutative[5] and satisfy laws of absorption, i.e., $a \cup (a \cap b) = a$ and $a \cap (a \cup b) = a$, for all a, b.

example: lattices
order-theoretic …
… and algebraic

Prima facie, order-theoretic lattices and algebraic lattices are two different types of structure, having nothing in common: an order-theoretic lattice can never be isomorphic to an algebraic lattice, because the former involves one dyadic relation whereas the latter involves two binary functions. Nevertheless mathematicians consider the resulting theories as just two different ways of characterizing one and the same type of structure. This is because the two theories are *definitionally equivalent*:[6]

definitional equivalence of theories

*eMail: `Christopher.von.Buelow@uni-konstanz.de`; Homepage: `www.uni-konstanz.de/FuF/Philo/Philosophie/Spohn/vonBuelow`

[1] See, e.g., Davey and Priestley 1990.

[2] I.e., a dyadic relation '$\leq$' such that for all a, b, c, the following hold: $a \leq a$ (reflexivity), $a \leq b \leq c \rightarrow a \leq c$ (transitivity), $a \leq b \wedge b \leq a \rightarrow a = b$ (antisymmetry).

[3] I.e., greatest lower bounds: objects $i_{ab} \in A$ such that $i_{ab} \leq a$, b and for all $i \in A$, if $i \leq a$, b then $i \leq i_{ab}$.

[4] I.e., least upper bounds, cf. fn. 3.

[5] I.e., $a \cup (b \cup c) = (a \cup b) \cup c$, $a \cap (b \cap c) = (a \cap b) \cap c$, $a \cup b = b \cup a$, and $a \cap b = b \cap a$, for all a, b, c.

[6] See Corcoran 1980, and Wilson 1981 (esp. p. 411), where the term 'interdefinable' is used instead. Cf. also Shapiro 1997, 241.

Let $Th_{\leq}$ and $Th_{\cap\cup}$ be the theories given by the axioms for order-theoretic lattices and for algebraic lattices, respectively; and consider the following definitions:

$$i = a \cap b \quad \leftrightarrow \quad i \leq a, b \ \wedge \ \forall j \leq a, b\colon \ j \leq i, \tag{$D_\cap$}$$

$$s = a \cup b \quad \leftrightarrow \quad s \geq a, b \ \wedge \ \forall t \geq a, b\colon \ t \geq s; \tag{$D_\cup$}$$

$$a \leq b \quad \leftrightarrow \quad a \cap b = a.^7 \tag{$D_\leq$}$$

(Here, $a \cap b$ and $a \cup b$ are defined to be the infimum and the supremum of $\{a, b\}$, respectively, whose uniqueness follows from the axioms. Reading '$\cap$' as 'infimum' in this way, a must obviously be $\leq$ b iff it is the infimum of the two.) Then, the order-theoretic axioms together with the definitions $(D_\cap)$ and $(D_\cup)$ imply both the algebraic axioms (and thus the whole theory $Th_{\cap\cup}$) and the definition $(D_\leq)$, and vice versa:

$$Th_{\leq} + (D_\cap) + (D_\cup) \ = \ Th_{\cap\cup} + (D_\leq).$$

In other words, there is a theory which is a definitional extension[8] (via $(D_\cap)$ and $(D_\cup)$, and via $(D_\leq)$) of both $Th_{\leq}$ and $Th_{\cap\cup}$.

nonisomorphic systems having the same structure

This implies that under the definitions given, there is for every model S_1 of the one theory a corresponding model S_2 of the other which has the same domain and, in a certain sense, the *same structure* (without being isomorphic to S_1): for every state of affairs in S_1 that can be expressed by a formula φ_1 of the one language, there is a corresponding formula φ_2 of the other language such that φ_1 and φ_2 can be shown to be equivalent on the basis of either set of axioms together with its supplementary definition(s); and φ_1 holds in S_1 (under some variable-assignment h) iff φ_2 holds in S_2 (under h).

… because of a common definitional expansion

The relationship which thus obtains between S_1 and S_2 can also be characterized as follows: Let $S_1 = \langle A, \leq_A \rangle$ and $S_2 = \langle A, \cap_A, \cup_A \rangle$, then $\bar{S} := \langle A, \leq_A, \cap_A, \cup_A \rangle$ is a definitional expansion[9] (via $(D_\cap)$ and $(D_\cup)$, and via $(D_\leq)$) of both S_1 and S_2:[10]

$$\begin{tikzcd}
& \langle A, \leq_A, \cap_A, \cup_A \rangle \arrow[dl, dash, "{(D_\cap),(D_\cup)}"'] \arrow[dr, dash, "(D_\leq)"] & \\
\langle A, \leq_A \rangle & & \langle A, \cap_A, \cup_A \rangle.
\end{tikzcd}$$

… because of isomorphic definitional expansions

What particular domain S_1 has is of course wholly irrelevant for its structure. So, for S_1 to have the same structure as some other system S_2, it should suffice if, instead of having a *common* definitional expansion, the two systems have *isomorphic* definitional expansions. This allows for their having different domains – or the same domain but with its objects permuted. Thus an order-theoretic lattice $\langle A, \leq_A \rangle$ and an algebraic one, $\langle B, \cap_B, \cup_B \rangle$, would have the same structure in this sense if the fol-

[7] On the right-hand side, '$a \cup b = b$' would do just as well.

[8] This is what Shoenfield (1967, 60) calls 'extension by definitions'.

[9] Or "expansion by definitions" (Shoenfield 1967, 134).

[10] That $\bar{S}$ is a definitional expansion of S_i corresponds to S_i's being a "full subsystem" of $\bar{S}$ (cf. Shapiro 1997, 91), and to S_i's being a "truncation" of $\bar{S}$ containing $\bar{S}$ as a "subpattern" (cf. Resnik 1981, 536; Resnik 1997, 209).

lowing situation obtained:

$$\begin{tikzcd}
\langle A, \leq_A, \cap_A, \cup_A \rangle \arrow[r, dash, "\cong"] \arrow[d, dash, "(D_\cap),(D_\cup)"'] & \langle B, \leq_B, \cap_B, \cup_B \rangle \arrow[d, dash, "(D_\leq)"] \\
\langle A, \leq_A \rangle & \langle B, \cap_B, \cup_B \rangle.
\end{tikzcd}$$

Generalizing from lattices to arbitrary systems, we get the following candidate for a concept of having the same structure (and thus for an identity criterion for structures):

structure equivalence

(I_1') S_1 and S_2 have the same structure iff S_1 and S_2 have definitional expansions $\bar{S}_1$ and $\bar{S}_2$ which are isomorphic.

This can be illustrated with the diagram

$$\begin{tikzcd}
\bar{S}_1 \arrow[r, dash, "\cong"] \arrow[d, dash, "D_1"'] & \bar{S}_2 \arrow[d, dash, "D_2"] \\
S_1 & S_2,
\end{tikzcd}$$

where D_1 and D_2 stand for sets of explicit definitions. This equivalence relation is called *structure equivalence* by Shapiro (1997, 91) and corresponds to Resnik's *pattern equivalence* (1997, 209; 1981, 536).[11] Structure equivalence is, as it were, isomorphism modulo definability. Structure equivalence is of course implied by isomorphism.

$\langle \mathbb{N}, \mathrm{Succ} \rangle$ and $\langle \mathbb{N}, +, 0, 1 \rangle$ and definability in higher-order logic

Structure equivalence is really not just one equivalence relation but rather a whole family of them. Consider the system of the natural numbers with the successor relation, $\langle \mathbb{N}, \mathrm{Succ} \rangle$, and the system of the natural numbers with addition, zero and 1, $\langle \mathbb{N}, +, 0, 1 \rangle$. There is a simple definition of successorship on the basis of addition and 1:[12]

$$n = \mathrm{Succ}\, m \quad \leftrightarrow \quad n = m+1, \tag{D_{Succ}}$$

and, going in the other direction, there are easy definitions of zero and 1 on the basis of successorship:

$$n = 0 \quad \leftrightarrow \quad \forall m\colon\; n \neq \mathrm{Succ}\, m, \tag{D_0}$$
$$n = 1 \quad \leftrightarrow \quad n = \mathrm{Succ}\, 0. \tag{D_1}$$

But successorship does not admit – at least not in the language of first-order logic – of an explicit definition of addition.[13] If, however, we allow ourselves to use second-order logic, we can give the following definition:

addition isn't 1st-order-definable from 'successor'

... but it is 2nd-order-definable

$$n = l + m \quad \leftrightarrow \quad \forall X^{(3)} \Big[\forall i\colon Xi0i \;\wedge\; \forall i,j,k \big(Xijk \rightarrow Xi\, \mathrm{Succ}\, j\, \mathrm{Succ}\, k \big) \;\rightarrow\; Xlmn \Big].^{14} \tag{D_+}$$

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[11] Resnik and Shapiro give characterizations which, though equivalent, amount to the more complicated diagram

$$\begin{tikzcd}
 & \bar{T} \arrow[dl, dash, "D_1"'] \arrow[dr, dash, "D_2"] & \\
S_1 \arrow[r, dash, "\cong"] & T_1 \quad\quad T_2 \arrow[r, dash, "\cong"] & S_2.
\end{tikzcd}$$

[12] In what follows, I use 'Succ' as a function symbol for the sake of brevity and readability.

[13] Given the particular system $\langle \mathbb{N}, \mathrm{Succ} \rangle$, we can characterize addition implicitly via the first-order recursion conditions $n+0 = n$ and $n + \mathrm{Succ}\, m = \mathrm{Succ}(n+m)$. These do not constitute an implicit definition in the sense of fixing, together with the theory of $\langle \mathbb{N}, \mathrm{Succ}, 0 \rangle$, the extension of '+' in arbitrary models. Thus Beth's Theorem cannot be exploited to infer the existence of an explicit definition.

n'th-order structure equivalence

So, whereas $\langle \mathbb{N}, \mathrm{Succ} \rangle$ and $\langle \mathbb{N}, +, 0, 1 \rangle$ are not structure-equivalent with respect to first-order logic (unlike the two lattices $\langle A, \leq_A \rangle$ and $\langle A, \cap_A, \cup_A \rangle$), they are so with respect to second-order logic. I define two systems S_1 and S_2 to be *n'th-order structure-equivalent* iff they have definitional expansions $\bar{S}_1$ and $\bar{S}_2$ with respect to n'th-order logic which are isomorphic. As isomorphism entails first-order structure equivalence, so n'th-order structure equivalence entails $(n+1)$'th-order structure equivalence. Thus we get a family of successively coarser-grained equivalence relations for systems and candidate identity criteria for structures ($n > 0$):

(I_n) S_1 and S_2 have the same structure iff S_1 and S_2 are n'th-order structure-equivalent.

what about "the" relation places of a structure?

By accepting structure equivalence, not isomorphism, as the adequate conception of sameness in structure, my suggestion that structures have relation and function places besides their object places (p. **??**) seems to be cast into doubt. If the lattices $\langle A, \leq_A \rangle$ and $\langle A, \cap_A, \cup_A \rangle$, being first-order structure-equivalent, have the same structure, $\mathfrak{S}$, then does $\mathfrak{S}$ have one relation place or rather two function places? There doesn't seem to be any definite array of relation (and function) places which belongs to $\mathfrak{S}$ itself; rather, each type of systems exemplifying $\mathfrak{S}$ has its own array of relation places: one for order-theoretic lattices isomorphic to $\langle A, \leq_A \rangle$, another for algebraic lattices isomorphic to $\langle A, \cap_A, \cup_A \rangle$, still another for their definitional expansions like $\langle A, \leq_A, \cap_A, \cup_A \rangle$, etc.

a place for every definable relation

But there is another road open as well. If n'th-order structure equivalence is taken as the criterion for sameness in structure then we could also consider $\mathfrak{S}$ as having a relation place for *every* relation definable in n'th-order logic from whatever has been taken as basic in the characterization given for $\mathfrak{S}$. Thus $\mathfrak{S}$, the structure of $\langle A, \leq_A \rangle$, would, over and above its $\leq$-place, have a $\geq$-place to be occupied by the converse of the $\leq$-place's occupant, $\cap$- and $\cup$-places for the corresponding binary operations, and so on. Even the cardinal structures would each have at least one relation place, viz., the one to be occupied by their respective identity relations. If we restrict ourselves to first-order logic then $\langle \mathbb{N}, \mathrm{Succ} \rangle$ and $\langle \mathbb{N}, +, 0, 1 \rangle$ have different structures, with the relation places of the former system's structure constituting, or at least corresponding to, a proper subset of those of the latter system's structure; if, however, we adopt higher-order logic, these differences are nullified.

So, the concepts of relation, function, and distinguished-object places of structures shouldn't be abandoned. One must merely keep in mind that what relation (etc.) places a structure has depends on which notion of structure one employs, i. e., which logic one has chosen; and that there may be more relation places than meet the eye.

which identity criterion to choose?

Which identity criterion for structures is the *right* one? Mathematical practice strongly suggests that criterion (I_0), based on isomorphism, is not it. Also, criterion (I_1), based on first-order structure equivalence, seems still too restrictive. In any case, I see as yet no reason for us to choose. What is important, rather, is that we be clear that different notions of structure exist ('iso-structure', 'first-order structure', 'second-order structure', etc.), and that, if necessary, we specify which of these we use at any one moment.

additional definable *objects*?

Now, a notion of structure which permits us to add or delete definable relations

[14]Here, the superscript in '$X^{(3)}$' indicates that 'X' is to be a three-place variable. One could also use a two-place variable, since l is the only i we need on the right-hand side: $n = l + m \;\leftrightarrow\; \forall X^{(2)} [X0l \wedge$

in a system without thereby changing its structure may seem very liberal. What more could one desire? I believe the structure concept could have still more latitude, namely, it could allow for extensions and truncations of the domain by "definable" objects.[15] This desire is best motivated by taking a look at two ontological theories.

→ motivation from ontology: **realism** about universals …

Realists about universals maintain that besides particulars, i. e., ordinary objects, there also exist universals, i. e., properties and relations, which can be exemplified by particulars. *Nominalists*, on the other hand, believe there are no universals, only particulars. The *trope-theoretical* variety of nominalism, however, contends that in addition to ordinary particulars there are also nonordinary particulars, tropes, which are somewhat like particularized universals: Instead of all red things exemplifying one single universal, redness, each red thing has its own particular red-trope. Different red things do not have anything (any entity) in common; rather, their colour tropes are similar to each other, and dissimilar to, e. g., green-tropes.

… vs. trope-theoretical **nominalism**

Let's bring these competing ontological theories into mathematical form by conceiving of their respective pictures of the world like mathematical systems. The world as envisioned by the realist is a system $\langle D_R, \mathcal{U}, \mathcal{E}\rangle$, where the domain D_R contains what exists according to the realist, viz., ordinary particulars and universals. The universals are picked out by the monadic relation $\mathcal{U}$, and the dyadic relation $\mathcal{E}$ specifies which particular, i. e., non-universal, exemplifies which universal. – The world of the trope-theoretical nominalist is a system $\langle D_T, \mathcal{T}, \mathcal{H}, \mathcal{S}\rangle$, with D_T inhabited by ordinary particulars and tropes. The latter are distinguished from the former via the monadic relation $\mathcal{T}$, the dyadic relation $\mathcal{H}$ specifies which ordinary particular has which trope, and $\mathcal{S}$ is an equivalence relation that holds exactly between those tropes which are similar to each other.

Realists and nominalists alike asseverate that these two ontological theories are contrary to each other: they cannot both be true. Universals either exist or they don't, and likewise for tropes. Many authors take these putative differences seriously, but I find it impossible to do so. To me these theories look rather like two different ways of describing the same world structure, just as talking about order-theoretic lattices and talking about algebraic lattices are merely two different ways of talking about a single type of (first-order) structure.

serious opposition …

… or just superficial variants?

Even though in the specification of the trope-theoretical system $\langle D_T, \mathcal{T}, \mathcal{H}, \mathcal{S}\rangle$ universals and exemplification are nowhere mentioned explicitly, nevertheless the system in some sense contains them implicitly, by containing all the information necessary to reconstruct them. Take as "universals" the equivalence classes of tropes with respect to similarity; i. e., for each $t \in \mathcal{T}$ let $[t]_{\mathcal{S}} = \{ t' \in \mathcal{T} \mid t' \mathcal{S}\, t \}$, and define (the set of) universals for trope theory by

hidden universals in the trope-theoretical system

$$\mathcal{U}_T := \mathcal{T}/\mathcal{S} = \left\{ [t]_{\mathcal{S}} \,\middle|\, t \in \mathcal{T} \right\}.$$

Then define trope-theoretical exemplification by letting the ordinary particular x exemplify an ersatz universal $[t]_{\mathcal{S}}$ iff x has a trope similar to t; i. e., for all $x \in D_T \backslash \mathcal{T}$ and $t \in \mathcal{T}$:

$$x\ \mathcal{E}_T\ [t]_{\mathcal{S}} \quad \text{iff} \quad \text{there is a } t' \in [t]_{\mathcal{S}} \text{ such that } x\ \mathcal{H}\ t'.$$

Similarly, the realist system implicitly contains tropes. All we have to do to see

hidden tropes in the realist system

$\forall j, k\big(Xjk \to X \mathrm{Succ}\, j\, \mathrm{Succ}\, k\big) \to Xmn\big]$.

[15] This notion of object definability is not the ordinary one. In the ordinary sense, an object is definable iff it is definable as a distinguished object, i. e., iff a monadic relation true of only this object can be defined. In this sense, zero and 1 are definable in $\langle \mathbb{N}, \mathrm{Succ}\rangle$ (p. 3). This doesn't affect the domain since these objects were in it beforehand.

this is "index" universals u with particulars x exemplifying them, i. e., define (the set of) the realist's tropes by

$$\mathcal{T}_R := \{ \langle x, u \rangle \in D_R \times \mathcal{U} \mid x \notin \mathcal{U} \text{ and } x \mathcal{E} u \}.$$

Define having of these surrogate tropes by stipulating that, for ordinary particulars $x, y \in D_R \setminus \mathcal{U}$ and universals $u \in \mathcal{U}$,

$$x \mathcal{H}_R \langle y, u \rangle \quad \text{iff} \quad x = y.$$

Finally, define "tropes" to be similar to each other iff the universals they contain as their second components are identical:

$$\langle x, u \rangle \mathcal{S}_R \langle y, u' \rangle \quad \text{iff} \quad u = u'.$$

natural bijections between domains

Thus we get natural bijections between the objects of the one theory and constructions from the objects of the other theory: to a universal (say, redness) there corresponds a certain $\mathcal{S}$-equivalence class of tropes (the set of red-tropes); to a (red-) trope there corresponds a certain indexed universal, i. e., a pair consisting of a particular and a universal exemplified by it (say, the oldest fire engine in Constance, and redness); and the ordinary particulars are anyway the same in both theories. Building on this bijection we then get a factuality-preserving correspondence between states of affairs in the two systems.

unexemplified universals?

There is one potential stumbling block: the procedure as sketched will only yield such a bijection if all universals are exemplified. If unicornhood is a universal then it is an unexemplified one, and in the absence of unicorns, neither are there unicornhood-tropes to correspond to the universal. So, do these two theories describe different world structures after all? I will deal with this problem later on; for the nonce let us assume that universals are always exemplified.

"definitional equivalence"? – 1st attempt: 1-sorted

axioms for realism

The structural similarity between the realist theory and the trope theory looks somewhat like definitional equivalence (p. 1). Let us see how far we get by trying to prove this. The axioms for realist systems $\langle D_R, \mathcal{U}, \mathcal{E} \rangle$ would have to be the following.[16] Every universal is exemplified:

$$\mathcal{U}u \rightarrow \exists x\, x \mathcal{E} u; \tag{R$_1$}$$

exemplification $\mathcal{E}$ is always of universals by particulars, i. e., by non-universals:

$$x \mathcal{E} u \rightarrow \neg \mathcal{U}x \wedge \mathcal{U}u; \tag{R$_2$}$$

and there *are* universals:

$$\exists u\, \mathcal{U}u. \tag{R$_3$}$$

Together, these axioms imply that there are particulars: there is a universal (R_3), it is exemplified (R_1), and what exemplifies it must be a particular (R_2).

axioms for trope theory

The axioms for trope-theoretical systems $\langle D_T, \mathcal{T}, \mathcal{H}, \mathcal{S} \rangle$, on the other hand, would be as follows. For every trope t, there is exactly one object which has t:

$$\mathcal{T}t \rightarrow \exists! x\, x \mathcal{H} t; \tag{T$_1$}$$

similarity $\mathcal{S}$ is an equivalence relation:

$$t \mathcal{S} t \wedge (s \mathcal{S} t \rightarrow t \mathcal{S} s) \wedge (r \mathcal{S} s \wedge s \mathcal{S} t \rightarrow r \mathcal{S} t); \tag{T$_2$}$$

[16]For the sake of readability, I use the letters '$\mathcal{U}$' and '$\mathcal{E}$' (and '$\mathcal{T}$', '$\mathcal{H}$', '$\mathcal{S}$') in the object-language, too.

"having" $\mathcal{H}$ is always of tropes by ordinary particulars, i.e., by non-tropes:

$$x\mathcal{H}t \rightarrow \neg\mathcal{T}x \wedge \mathcal{T}t; \tag{T$_3$}$$

similarity is only between tropes:

$$s\mathcal{S}t \rightarrow \mathcal{T}s \wedge \mathcal{T}t; \tag{T$_4$}$$

and there are tropes:

$$\exists t\, \mathcal{T}t. \tag{T$_5$}$$

Like before, these imply that there are (ordinary) particulars.

Now we have to find definitions for the concepts of each theory's complement. The task is easier in the case of trope theory. "Universals" are $\mathcal{S}$-equivalence classes, i.e., certain sets of tropes, so we define $\mathcal{U}_T$ to be the monadic second-order relation which applies to a set $U \subset D_T$ iff there is a trope t such that U is the set $[t]_{\mathcal{S}}$ of those tropes which are similar to t:

definitions of $\mathcal{U}$ and $\mathcal{E}$ in trope theory

$$\mathcal{U}_T U^{(1)} \quad \leftrightarrow \quad \exists t\big[\mathcal{T}t \wedge \forall s(Us \leftrightarrow s\mathcal{S}t)\big]. \tag{D$_\mathcal{U}$}$$

The objects s in U must then be tropes because of axiom (T$_4$). "Exemplification" is a mixed first/second-order relation between particulars x on the one hand and "universals" U on the other which obtains iff x has a trope in U:

$$x\mathcal{E}_T U^{(1)} \quad \leftrightarrow \quad \mathcal{U}_T U \wedge \exists t(Ut \wedge x\mathcal{H}t). \tag{D$_\mathcal{E}$}$$

In this case x must be an ordinary particular because of axiom (T$_3$).

Proceeding to realism, if we try to represent tropes as pairs, as we did in the preceding informal account, and use for this purpose the Kuratowski definition $\langle a, b\rangle := \big\{\{a\},\{a,b\}\big\}$, then the property of being a "trope" $\langle x, u\rangle$ turns out to be third order,[17] and rather complicated besides. In the context of our formalized realist theory, however, we can make do with second-order logic, by defining "tropes" to be sets containing (exactly) two objects x, u such that x exemplifies u:

definitions of $\mathcal{T}$, $\mathcal{H}$ and $\mathcal{S}$ for realists

$$\mathcal{T}_R T^{(1)} \quad \leftrightarrow \quad \exists x, u\big[x\mathcal{E}u \wedge \forall z(Tz \leftrightarrow z{=}x \vee z{=}u)\big]. \tag{D$_\mathcal{T}$}$$

Axiom (R$_2$) then guarantees that x is a particular and u is a universal. The "having" of these "tropes" can be defined correspondingly as membership of a particular in such a "trope":

$$x\mathcal{H}_R T^{(1)} \quad \leftrightarrow \quad \neg\mathcal{U}x \wedge \mathcal{T}_R T \wedge Tx. \tag{D$_\mathcal{H}$}$$

Finally, two "tropes" S and T are "similar" to each other iff they have a universal u as a common member, that is, iff the universals they contain are identical:

$$S^{(1)}\mathcal{S}_R T^{(1)} \quad \leftrightarrow \quad \mathcal{T}_R S \wedge \mathcal{T}_R T \wedge \exists u(\mathcal{U}u \wedge Su \wedge Tu). \tag{D$_\mathcal{S}$}$$

One could now go ahead and try to prove that (R$_1$)–(R$_3$), (D$_\mathcal{T}$), (D$_\mathcal{H}$), (D$_\mathcal{S}$) together imply (T$_1$)–(T$_5$), (D$_\mathcal{U}$), (D$_\mathcal{E}$), and vice versa, and thus demonstrate that the two theories are definitionally equivalent. This cannot work as straightforwardly as it did for lattices and for number theory, because here the analogues of some of the first-order objects of the one theory (say, the realist's universals) are second-order objects of the other (trope-theoretical "universals"). This means that we have

"definitional equivalence"? – translation problems!

[17]Sets of sets of objects are second-order "properties", i.e., third-order objects.

to translate parts of the axioms and definitions. For example, the realist axiom (R_1) $(\mathcal{U}u \to \exists x\ x\,\mathcal{E}\,u)$ would have to become '$\mathcal{U}_T U^{(1)} \to \exists x\ x\,\mathcal{E}_T\,U$' to be adequate for the formalized trope theory. This wouldn't constitute such a great obstacle if it weren't for the fact that some formulae do not have syntactically well-formed translations. If we perfunctorily translate axiom (R_2) $(x\,\mathcal{E}\,u \to \neg\mathcal{U}x \wedge \mathcal{U}u)$, we get '$x\,\mathcal{E}_T\,U^{(1)} \to \neg\mathcal{U}_T x \wedge \mathcal{U}_T U$'; but '$\mathcal{U}_T x$' makes no sense, since the predicate '$\mathcal{U}_T$' is reserved for second-order variables, variables for sets, not for first-order objects.[18] Analogous problems arise for attempts to translate, conversely, from trope-theoretical language to realist language.

"definitional equivalence"? – 2nd attempt: 2-sorted

realist systems and axioms

To avoid these problems it seems best to supplant the one-sorted languages we have hitherto used with two-sorted languages. Realist systems then have two domains: a nonempty set of (ordinary) particulars, P, and a nonempty set of universals, U. The variables 'x', 'y', 'z' (possibly with primes $'$ or with subscripts) are to get their values from within P, the variables 'u', 'v', 'w' (possibly with primes or subscripts), from within U. The job of differentiating between universals and particulars is thus done by these two kinds of first-order individual variables, rendering the predicate '$\mathcal{U}$' obsolete. We keep the relation symbol '$\mathcal{E}$', which must now be interpreted as a subset of $P \times U$. So, a realist system has the form $\langle P, U; \mathcal{E}\rangle$ (where the semicolon is used to separate domains from relations). As axioms we only need a modified version of (R_1) now:

$$\forall u\, \exists x\ x\,\mathcal{E}\,u, \tag{R_1'}$$

because what had to be stated explicitly in (R_2) and (R_3) on our former account is now contained implicitly in the language and its semantics.

trope-theoretical systems and axioms

Trope-theoretical systems similarly have two domains P and T, containing the ordinary particulars and the tropes, respectively. We use the variables 'x', 'y', 'z' (and variants thereof) for ordinary particulars again, and the variables 'r', 's', 't' (and variants), for tropes. Like '$\mathcal{U}$', the predicate '$\mathcal{T}$' is obsolete; and we can dispense with axioms (T_3)–(T_5), stipulating that $\mathcal{H} \subset P \times T$ and $\mathcal{S} \subset T^2$. Thus a trope-theoretical system $\langle P, T; \mathcal{H}, \mathcal{S}\rangle$ must merely satisfy the axioms

$$\forall t\, \exists! x\ x\,\mathcal{H}\,t \tag{T_1'}$$

and

$$t\,\mathcal{S}\,t \ \wedge\ (s\,\mathcal{S}\,t \to t\,\mathcal{S}\,s) \ \wedge\ (r\,\mathcal{S}\,s \wedge s\,\mathcal{S}\,t \to r\,\mathcal{S}\,t). \tag{T_2}$$

new definitions of $\mathcal{U}$ and $\mathcal{E}$ for trope theory

The definitions we need for trope theory are almost as before (slightly simpler, in the case of $(D'_{\mathcal{U}})$); I use the superscript '$^{(T)}$' instead of '$^{(1)}$' to indicate that the second-order variable 'U'[19] is monadic with arguments from the set T of tropes.

$$\mathcal{U}_T U^{(T)} \ \leftrightarrow\ \exists t\, \forall s (U s \leftrightarrow s\,\mathcal{S}\,t); \tag{$D'_{\mathcal{U}}$}$$

$$x\,\mathcal{E}_T\,U^{(T)} \ \leftrightarrow\ \mathcal{U}_T U \wedge \exists t (U t \wedge x\,\mathcal{H}\,t). \tag{$D'_{\mathcal{E}}$}$$

new definitions of $\mathcal{T}$, $\mathcal{H}$ and $\mathcal{S}$ for realism

As for realism, it seems cleaner to conceive of "tropes" not as properties, which would have to be domain-spanning now, but rather as relations between a unique

[18] Nor does it help to introduce an additional predicate '$\mathcal{P}$' for 'particular', and use '$\mathcal{P}x$' instead of '$\neg\mathcal{U}x$' in (R_2) and $(D_{\mathcal{H}})$: we would again have theorems like $\forall z(\mathcal{P}z \vee \mathcal{U}z)$, which aren't usefully translatable.

[19] I apologize for more ambiguities to be resolved by context: whereas 'U' stands for the set of universals in the context of realism, it is a second-order variable in the context of trope theory; analogously for 'T'.

particular and a unique universal, "arrows" from P to U, as it were:[20]

$$\mathcal{T}_{\mathrm{R}}\mathsf{T}^{(\mathsf{P},\mathsf{U})} \leftrightarrow \exists x,u\big[x\,\mathcal{E}\,u \wedge \forall y,v(\mathsf{T}yv \leftrightarrow y=x \wedge v=u)\big]; \qquad (\mathrm{D}'_{\mathcal{T}})$$

$$x\,\mathcal{H}_{\mathrm{R}}\mathsf{T}^{(\mathsf{P},\mathsf{U})} \leftrightarrow \mathcal{T}_{\mathrm{R}}\mathsf{T} \wedge \exists u\,\mathsf{T}xu; \qquad (\mathrm{D}'_{\mathcal{H}})$$

$$\mathsf{S}^{(\mathsf{P},\mathsf{U})}\,\mathcal{S}_{\mathrm{R}}\,\mathsf{T}^{(\mathsf{P},\mathsf{U})} \leftrightarrow \mathcal{T}_{\mathrm{R}}\mathsf{S} \wedge \mathcal{T}_{\mathrm{R}}\mathsf{T} \wedge \exists x,y,u(\mathsf{S}xu \wedge \mathsf{T}yu). \qquad (\mathrm{D}'_{\mathcal{S}})$$

Before we proceed to prove the "definitional equivalence" of the two theories, it will be helpful to specify the translations that are to be employed. The realist language we are dealing with is supposed to be talking about ordinary particulars, universals and "tropes" only, not about arbitrary sets, relations, or functions on the two domains P and U. This means that second-order variables 'R', 'S', 'T', ... only appear in quantifications of the forms $\forall\mathsf{T}^{(\mathsf{P},\mathsf{U})}\big[\mathcal{T}_{\mathrm{R}}\mathsf{T} \rightarrow \varphi(\mathsf{T})\big]$ and $\exists\mathsf{T}^{(\mathsf{P},\mathsf{U})}\big[\mathcal{T}_{\mathrm{R}}\mathsf{T} \wedge \varphi(\mathsf{T})\big]$, with '$\mathsf{T}$' free in $\varphi(\mathsf{T})$. I designate the trope-theoretical translation of a realist formula φ by '${}^{\mathsf{T}}\varphi$'. The translation can thus be determined by the following recursion clauses:

translations

translation from realist language to trope-theoretical

$$\begin{aligned}
{}^{\mathsf{T}}x=y &:= x=y,\\
{}^{\mathsf{T}}u=v &:= \mathsf{U}^{(\mathsf{T})}=\mathsf{V}^{(\mathsf{T})},\\
{}^{\mathsf{T}}\mathsf{S}=\mathsf{T} &:= s=t,\\
{}^{\mathsf{T}}x\,\mathcal{E}\,u &:= x\,\mathcal{E}_{\mathsf{T}}\mathsf{U}^{(\mathsf{T})},\\
{}^{\mathsf{T}}x\,\mathcal{H}_{\mathrm{R}}\mathsf{T} &:= x\,\mathcal{H}\,t,\\
{}^{\mathsf{T}}\mathsf{S}\,\mathcal{S}_{\mathrm{R}}\mathsf{T} &:= s\,\mathcal{S}\,t,\\
{}^{\mathsf{T}}\mathsf{T}xu &:= x\,\mathcal{H}\,t \wedge \mathsf{U}t, \qquad (*)\\
{}^{\mathsf{T}}\neg\varphi &:= \neg{}^{\mathsf{T}}\varphi,\\
{}^{\mathsf{T}}(\varphi\wedge\psi) &:= {}^{\mathsf{T}}\varphi \wedge {}^{\mathsf{T}}\psi,\\
&\ \vdots\\
{}^{\mathsf{T}}\forall x\ \varphi(x) &:= \forall x\ {}^{\mathsf{T}}\varphi(x),\\
{}^{\mathsf{T}}\exists x\ \varphi(x) &:= \exists x\ {}^{\mathsf{T}}\varphi(x),\\
{}^{\mathsf{T}}\forall u\ \varphi(u) &:= \forall\mathsf{U}^{(\mathsf{T})}\big[\mathcal{U}_{\mathsf{T}}\mathsf{U} \rightarrow {}^{\mathsf{T}}\varphi(\mathsf{U})\big],\\
{}^{\mathsf{T}}\exists u\ \varphi(u) &:= \exists\mathsf{U}^{(\mathsf{T})}\big[\mathcal{U}_{\mathsf{T}}\mathsf{U} \wedge {}^{\mathsf{T}}\varphi(\mathsf{U})\big],\\
{}^{\mathsf{T}}\forall\mathsf{T}^{(\mathsf{P},\mathsf{U})}\big[\mathcal{T}_{\mathrm{R}}\mathsf{T}\rightarrow\varphi(\mathsf{T})\big] &:= \forall t\ {}^{\mathsf{T}}\varphi(t),\\
{}^{\mathsf{T}}\exists\mathsf{T}^{(\mathsf{P},\mathsf{U})}\big[\mathcal{T}_{\mathrm{R}}\mathsf{T}\wedge\varphi(\mathsf{T})\big] &:= \exists t\ {}^{\mathsf{T}}\varphi(t).
\end{aligned}$$

The formula '$\mathsf{T}xu$' in $(*)$, which somewhat artificially mixes universals and tropes, must nevertheless be accounted for; it is best read as: 'T is the u-trope of x' – like, e.g., the fire engine's colour-trope is a red-trope. Analogously, the formula '$\mathsf{U}t$' – see (†) below – of the extended trope-theoretical language can be read as: 't is a U-trope.' This explains the way they appear in each other's translations.

Matters are similar for the translation of trope-theoretical formulae φ to realist formulae ${}^{\mathrm{R}}\varphi$:

translation from trope-theoretical language to realist

$$\begin{aligned}
{}^{\mathrm{R}}x=y &:= x=y,\\
{}^{\mathrm{R}}s=t &:= \mathsf{S}^{(\mathsf{P},\mathsf{U})}=\mathsf{T}^{(\mathsf{P},\mathsf{U})},
\end{aligned}$$

[20] The superscript '${}^{(\mathsf{P},\mathsf{U})}$' indicates that the first argument is to be from P, and the second, from U.

$$\begin{aligned}
{}^{\mathrm{R}}U=V &:= u=v,\\
{}^{\mathrm{R}}x\,\mathcal{H}\,t &:= x\,\mathcal{H}_{\mathrm{R}}\,T^{(P,U)},\\
{}^{\mathrm{R}}s\,\mathcal{S}\,t &:= S^{(P,U)}\,\mathcal{S}_{\mathrm{R}}\,T^{(P,U)},\\
{}^{\mathrm{R}}x\,\mathcal{E}_{T}U &:= x\,\mathcal{E}\,u,\\
{}^{\mathrm{R}}Ut &:= \exists x\ Txu, && (\dagger)\\
{}^{\mathrm{R}}\neg\varphi &:= \neg{}^{\mathrm{R}}\varphi,\\
{}^{\mathrm{R}}(\varphi\wedge\psi) &:= {}^{\mathrm{R}}\varphi\wedge{}^{\mathrm{R}}\psi,\\
&\ \ \vdots\\
{}^{\mathrm{R}}\forall x\ \varphi(x) &:= \forall x\ {}^{\mathrm{R}}\varphi(x),\\
{}^{\mathrm{R}}\exists x\ \varphi(x) &:= \exists x\ {}^{\mathrm{R}}\varphi(x),\\
{}^{\mathrm{R}}\forall t\ \varphi(t) &:= \forall T^{(P,U)}\big[\mathcal{T}_{\mathrm{R}}T\rightarrow{}^{\mathrm{R}}\varphi(T)\big],\\
{}^{\mathrm{R}}\exists t\ \varphi(t) &:= \exists T^{(P,U)}\big[\mathcal{T}_{\mathrm{R}}T\wedge{}^{\mathrm{R}}\varphi(T)\big],\\
{}^{\mathrm{R}}\forall U^{(T)}\big[\mathcal{U}_{T}U\rightarrow\varphi(U)\big] &:= \forall u\ {}^{\mathrm{R}}\varphi(u),\\
{}^{\mathrm{R}}\exists U^{(T)}\big[\mathcal{U}_{T}U\wedge\varphi(U)\big] &:= \exists u\ {}^{\mathrm{R}}\varphi(u).
\end{aligned}$$

Now we can go ahead and prove (a) the *translations* ${}^{\mathrm{T}}(\mathrm{R}_1')$, ${}^{\mathrm{T}}(\mathrm{D}_{\mathcal{T}}')$, ${}^{\mathrm{T}}(\mathrm{D}_{\mathcal{H}}')$, ${}^{\mathrm{T}}(\mathrm{D}_{\mathcal{S}}')$ of the realist axioms and definitions on the basis of the trope-theoretical axioms and definitions, and (b) the translations ${}^{\mathrm{R}}(\mathrm{T}_1')$, ${}^{\mathrm{R}}(\mathrm{T}_2)$, ${}^{\mathrm{R}}(\mathrm{D}_{\mathcal{U}}')$, ${}^{\mathrm{R}}(\mathrm{D}_{\mathcal{E}}')$ of the trope-theoretical axioms and definitions, on the basis of the realist axioms and definitions. The proofs themselves are easy and uninteresting.[21]

proof of ${}^{\mathrm{R}}(\mathrm{T}_1')$ in the realist setting

As an example, I present an informal proof of the translated first axiom of trope theory, ${}^{\mathrm{R}}(\mathrm{T}_1')$, i.e.,

$$\forall T^{(P,U)}(\mathcal{T}_{\mathrm{R}}T\ \rightarrow\ \exists!x\ x\,\mathcal{H}_{\mathrm{R}}T),$$

in the realist setting: Let $T^{(P,U)}$ be arbitrary (i.e., an arbitrary subset of $P\times U$) with $\mathcal{T}_{\mathrm{R}}T$. According to $(\mathrm{D}_{\mathcal{T}}')$, this means:

$$\exists x,u\big[x\,\mathcal{E}\,u\ \wedge\ \forall y,v(Tyv\ \leftrightarrow\ y=x\wedge v=u)\big]. \tag{1}$$

Let x and u be fixed; reading the biconditional from right to left we get Txu, i.e., T is the u-trope of x. The two propositions $\mathcal{T}_{\mathrm{R}}T$ and Txu together imply $x\,\mathcal{H}_{\mathrm{R}}T$ (see $(\mathrm{D}_{\mathcal{H}}')$): x has T; so all that is left to show is that x is the *only* ordinary particular which has T. Let y be arbitrary with $y\,\mathcal{H}_{\mathrm{R}}T$. By $(\mathrm{D}_{\mathcal{H}}')$, this entails that there is a universal v such that Tyv, and, reading the biconditional in (1) from left to right now, we derive that $y = x$ – which completes the proof.

We can now see the extended sense in which these two ontological theories are definitionally equivalent, and also the extended sense in which additional objects may be definable in a system. And finally we get a notion of "having the same structure" which is even more liberal than structure equivalence – or is it??

[21] In some places one has to make use of the axiom schema of comprehension to secure the existence of certain second-order entities.

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