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25 Quantity Has a Quality All Its Own

Abstract: Since the work of Cantor, philosophical reflection arguments have been developed as justification for basic set-theoretic principles that are seen as mathematical reflection principles and that postulate large actual infinities of mathematical objects. A main objective of this paper is to discuss philosophical arguments for such ontological reflection principles, and their roots in the history of philosophy and theology.

Just don't get ontological. Not now.
I couldn't bear it if you were ontological with me.
(Mr. Big, in: Allen 1991, p. 288)

25.1 Introduction

In this article, I explore aspects of the relation between quantitative infinity on the one hand, and a particular concept of reflection on the other hand. The focus will lie exclusively on a particular form of ontological reflection. Epistemic concepts of reflection have also played a role in the history of philosophy (for instance in the work of Locke and Leibniz), but they will not be treated in this article.

We will see that ontological reflection has deep roots in the history of philosophy and theology. Ontological reflection first came to the fore in late Antiquity, in discussions about the nature of God. The connection between ontological reflection on the one hand, and quantitative infinity on the other hand, became clear only much later: from the late nineteenth century onward. At around this time, the influence of theology as an intellectual discipline was receding rapidly. The relative scientific influence of philosophy was also declining, albeit less rapidly.

Reflection considerations play an important role in contemporary set theory. On the one hand, what are loosely called "reflection arguments" play a role in mathematical proofs in set theory. On the other hand, since the work of Cantor, *philosophical* reflection arguments have been adduced as justification for basic set-theoretical principles that are seen as mathematical reflection axioms. A main

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objective of the present article is to discuss these philosophical reflection arguments, and their roots in the history of philosophy and theology, in some detail.

Before we start off, let me express a disclaimer. Thoroughly to investigate the relations between ontological reflection across three disciplines (philosophy, theology, set theory) requires a breath of expertise that very few people can claim to have, and that I certainly do not claim to possess. I can only express my hope that others will in the future explore these relations further.

25.2 From Myth to Philosophy

Creation myths attempt metaphorically to explain where the order in our world comes from. They may, for instance, relate the struggle between a primeval animal- or monster-like entity on the one hand, and a super-human agent on the other hand, which ends with the super-human agent defeating the monster, and thereby imposing order on the universe.

Pre-Socratic Greek philosophers of nature transformed creation myths into rational explanations, and thereby sought to acquire rational understanding of the nature of our world as a law-governed *cosmos*. This transformation involved a de-anthropomorphization of the creation myths.

Even though our knowledge of his philosophy is extremely fragmentary, it is clear that Anaximander is a key intermediary figure in this transformation process. Anaximander replaces the mythical pre-order state by a philosophical principle that he calls the *Apeiron*. Literally, Apeiron means something like "without boundary/end/limit," "what is not fenced in." In its philosophical use, Apeiron is variously translated as indefinite, indeterminate, unlimited, formless. (The distinctions between these different notions had not crystalized in Anaximander's days.) According to Anaximander, out of the Apeiron, all distinctions in our world (such as the "elements" water, fire, etcetera) are somehow generated, and to the Apeiron they will eventually return. We will see how Anaximander's Apeiron is the far-distant source of our modern concept of infinity.

According to Aristotle's *hylemorphic* metaphysics, every object consists of form impressed on matter. A form is a determination, a "fence," if you like. In *Metaphysics*, vii, 3, Aristotle describes formless matter in the following terms:

> By "matter" I mean that which in itself is not called a substance nor a quantity nor anything else by which being is categorised. For it is something of which each of these things is predicated, whose being is different from each of its predicates (for the others are predicated of substance, and substance is predicated of matter). Therefore this last is in itself neither substance nor quantity nor anything else. Nor is it the denials of any of these; for even denials belong to things accidentally. (Aristotle, 2016, 1029a20–26)

Thus formless matter shares many properties with Anaximander's Apeiron: it is in no way "fenced in." Pure matter also seemingly cannot be known as it is in itself, since knowledge necessarily involves subsumption under categories, which are conceptual representations of forms. This then immediately gives rise to an epistemological puzzle, for in the above quotation, for instance, Aristotle presumably does purport to express knowledge about formless matter.

25.3 The Problem of Mathematics

In most of Ancient philosophy—Anaximander was an exception to this—as in the creation myths, the chaotic and formless has strong negative connotations, whereas the orderly and definite has strong positive connotations. In Greek philosophy, for this reason, the Apeiron was excluded, as far as possible, from the world.

As far as the physical world is concerned, the project of banning the unbounded from the world was relatively successful. Unlike Anaximander, Aristotle believed that matter cannot exist independently of form; Greek astronomy adopted the position that the cosmos is a *finite* globe.

In mathematics, problems seemed to arise. Aristotle's contemporary Euclid taught that a line segment consists of points, and the multiplicity of points of which a line segment consists cannot be bounded by a number. Arithmetic posed problems, too. For every natural number, there is a greater number, so the multitude of numbers cannot itself be numbered.

In response to these difficulties, Aristotle developed his distinction between potential infinity and actual infinity. An entity (object, quantity, …) is potentially infinite if it is necessarily finitely bounded, but at the same time *can* exceed any given finite bound. An entity is actually infinite if it *actually* exceeds all finite bounds. In this notion of potential infinity, a modality ("necessarily," "can") is involved. This modality can be given various interpretations: a temporal modality may be intended (always, sometimes), logical necessity/possibility may be meant, metaphysical necessity/ possibility may be what is relevant… At any rate, Aristotle's distinction quickly became the template for thinking about infinity in mathematics and in physics.

His distinction neatly solves the problem of the infinity of the natural numbers. According to Aristotle, the collection of natural numbers is potentially infinite. In any situation, only finitely many natural numbers exist. Thus the number of natural numbers can always be numbered and is thereby necessarily "fenced in." But there could always be more natural numbers than there are. The problem of line segments consisting of points is perhaps less convincingly solved by this strategy. A finite line segment *ab* can always be divided up into smaller line segments. But there can be no "infinite stage" at which, after at each finite stage choosing the left half of the remaining segment, the point *a* is finally reached. Thus "points" can be approached to arbitrary finite precision, but, in geometric reality, there are not and cannot be points. (So Euclidean geometry talks about certain entities that cannot exist!)

Aristotle' distinction also helps us with keeping the Apeiron out of our physical world. For every moment in time, there will be a later moment; but at no moment, an infinite period of time has elapsed. Thus also in the temporal direction, the cosmos is always finitely bounded. Every finite time interval can be divided in two; but there is no possible situation in which a time interval has been divided infinitely many times so that an unextended moment of time has been reached. Also, it seemed that Aristotle's distinction could be used to resolve Zeno's paradoxes, most of which seemed to revolve around the concept of infinity.

25.4 Mr. Big

In late Antiquity, two monotheistic religions, Judaism and Christianity, began to exercise a profound influence on Western thought. Philosophy at this time was in the fortunate and rare situation of important new data coming in. The following are among propositions that were increasingly seen as just as evident as immediate observation reports and elementary logical truths:
– There is one and only one God;
– God is perfect in every conceivable way;
– God transcends our intellectual powers.

This does not mean that in earlier Greek philosophy the hypothesis of a single God, for instance, was not somehow entertained. (Think of Aristotle's *Unmoved Mover*.) But it would have been treated as metaphysical speculation rather than as certainty. The reason that principles such as the above came in late Antiquity to be regarded as rock-solid *givens* of course has to do with the fact that they can be found in the Bible.

Concerning the physical world and the mathematical world, the dominant philosophical view remained substantially unchanged: it had no place for the Apeiron. There also was no *direct* pressure to associate God with infinity: the infinity of God is not mentioned in the Bible. Yet, the above principles entail that God exceeds all bounds. At some point the conceptual pressure became too great, and philosophers-theologians—the distinction between the two was often not clear—started to predicate infinity of God.[1] This precipitated a reverse of normative polarity concerning "bounded" and of "unbounded," like a switch of the magnetic poles of the earth that is said sometimes to occur. Infinity came to be seen as a *positive* attribute (a "good thing to have"), and boundedness as rather a negative property. It seems that Plotinus is an important intermediate figure in this process of reverse of polarity of infinity (Krainer, 2019, pp. 25–27). On the one hand, he predicates infinity of the One, and intends this as a positive attribution. On the other hand, he also predicates infinity of Evil, and in this case infinity is surely a negative property.

Aristotle's distinction was of no use in "taming" the infinity of God. Every unactualized potentiality in God would be an imperfection. There can then be no unactualized potentiality in God. And this means that if God is unbounded at all, then He is *actually* unbounded. The application of the concept of infinity primarily to God has brought about a momentous shift in the concept of infinity. Whereas before, pure matter, or, in Aristotelian terms, pure potentiality was the paradigmatic instance of infinity, now pure actuality becomes the paradigmatic instance of infinity. This reveals an interesting duality between the notions of pure potentiality and pure act.

Aristotle used the concept of infinity almost exclusively in a quantitative sense. When the concept of infinity comes in late Antiquity also to be applied to God, infinity comes to be applied to properties of God that are not easily conceived of in quantitative terms, such as goodness. What was *meant* when infinity was predicated of God, varied considerably, and was mostly a sense of qualitative infinity. The lesson of the modern theory of infinity—see the title of this paper—was not appreciated at this time.

The question about God's infinity became an important chapter in a long and extensive philosophico-theological debate about the nature of God. When they spoke of God's infinity, some authors had unbounded perfection in mind, sometimes a form of absolute simplicity was meant, at other times a combination of both. Indeed, the alleged combination of immensity and maximal simplicity of God posed a formidable challenge to attempts to construct a convincing theory of the infinity of God. A form of *quantitative* infinity was mostly not what was intended, although that, too, played a role (if God is omnipresent, for instance). In this context, it should be kept in mind that important conceptual distinctions that we currently routinely make (such as between indefinability, incomprehensibility, semantic indeterminateness, ontological indeterminacy, unboundedness, indefinite extensibility…) were only to a very limited degree available then.

1 Gregory of Nyssa is an important figure in this evolution: see Achtner, 2011, Section 1.4.2.

25.5 Reflection

Since God came in the Middle Ages to be increasingly thought to exceed all bounds, many came to believe that this in particular holds for all *conceptual* bounds. A consequence of this would be that positive (i.e., negation-free) propositional truths do not hold of God, whereby positive propositional knowledge of God is impossible. This line of thought is known as *negative theology*. Nonetheless, some philosophers put forward positive hypotheses about the nature of God. Some of these are *reflection hypotheses*. We discuss two of those, and we take them in reverse chronological order.

Augustine's views of infinity did not remain stable throughout his theological career.[2] But his writings contain a thought concerning quantitative infinity that has proved to be remarkably prescient. Augustine's thought concerns the multiplicity of the natural numbers. This multiplicity forms, according to received wisdom in Antiquity and the Middle Ages, a potential infinity in the world. But in God's knowledge this multiplicity is *limited* in the sense that He can somehow assign a number to it, so that something that is infinite for us, is finite for God (Augustine, 1972, 12.18):

> The infinity of number[s], although there is no number for infinities of numbers, is yet not incomprehensible by Him of whose understanding there is no number. And thus, if what is comprehended in knowledge is made finite by the comprehension of this knowledge, then all infinity is in some ineffable way finite to God, for it is not incomprehensible to His knowledge.

Thus the infinity of natural numbers is somehow *reflected* in a bounded entity in God's thought. This passage is many centuries later lauded by Cantor as a prefiguration of his theory of transfinite numbers (Cantor, 1962, *Mitteilungen zur Lehre vom Transfiniten*, p. 402):

> More energetic and more perfectly as is done here by St. Augustin, the Transfinite cannot be justified and defended. [...] By asserting the total, intuitive perception of the set of natural numbers [in God's knowledge], St. Augustin recognises this collection at the same time *formaliter* as an actually infinite whole, as a *Transfinitum*, and we are compelled to follow him in this. [my translation]

Three centuries earlier, in *On Dreams*, Philo of Alexandria postulated an inverse reflection phenomenon (Philo of Alexandria, 1988, pp. 419–421):

> Thus in another place, when he had inquired whether He that is has a proper name, he came to know full well that He has no proper name, [the reference is to Exodus 6:3] and that whatever name anyone may use for Him he will use by licence of language; for it is not in the

2 See Drozdek, 2019.

> nature of Him that is to be spoken of, but simply to be. Testimony to this is also afforded by the divine response made to Moses' question whether He has a name, even "I am He that is (Exodus 3:14)". It is given in order that, since there are not in God things that man can comprehend, man may recognise His substance. To the souls indeed which are incorporeal and occupied in His worship it is likely that He should reveal himself as He is, conversing with him as a friend with friends; but to souls which are still in the body, giving Himself the likeness of angels, not altering His own nature, for He is unchangeable, but conveying to those who receive the impression of His presence a semblance in a different form, such that they take the image not to be a copy, but that original form itself.

This is a reflection phenomenon not from the world to God (as with Augustin), but from God to the world. An "angel" *reflects* the essence of God in the form of an image. But this angel-image is such a perfect copy that we cannot distinguish it from God in any way, so we humans tend to take such an "angel" to be God himself. Philo thus posits the following *reflection principle*: God is reflected, in the sense of being mirrored, in an entity in the world (an angel). We will see that reflection principles that have this structure, are theoretically very powerful: complexity of the reflected object can be deduced from such principles. In the quoted passage Philo also observes—he was clever indeed!—that this theory leads to a *semantic problem*. Since we cannot distinguish God from certain "angels," there is nothing we can do to ensure that the word "God" refers to God rather than to one of the angels. So, literally speaking, on Philo's view, God is unnameable.

An idea going back at least to Aristotle is that the most perfect kind of intellectual activity is *self-thought*.[3] The reason seems to be that when a mind thinks about an entity that is not herself, then her thought cannot be completely adequate to the object of thought, because the object of thought is then not numerically identical to her mind. But this problem dissolves when subject and object of knowledge coincide. From this it is only a small step to conclude that the most perfect thought (self-thought) is most perfectly produced by the most perfect entity (God).

Philo was of course familiar with this line of thought. We can connect it with his metaphysics in a way that he might well have been sympathetic with, by internalizing his reflection idea that we discussed above to God Himself, in the following way.

In Philo's metaphysics, there is a sense in which God is maximally simple *as well as* a sense in which God is maximally immense. Philo maintains that God in his Essence is absolutely simple, but at the same time He externalises Himself in what Philo calls the External Logos, which is truly immense.[4] The externalized

3 See Menn, 2012.

4 See Wolfson, 1947, Chapter IV, Section IV.

Logos contains an abstract blueprint of the whole material world, is in some sense also responsible for the creation of our world, and has an immeasurably complicated and intricate structure.

Perhaps God in its deepest Essence thinks himself by simply being, by simply coinciding with himself.[5] But concerning the External Logos, the situation is interestingly different. Philo maintains that there is an (abstract) *idea of the Logos* (Wolfson, 1947, pp. 213–214). Suppose the Logos contains such an idea (call it *idea 1*) beside the abstract "blueprints" of all the objects in our world, so that the Logos is reflected *in itself*. Suppose furthermore that this idea is perfect, in the sense that it is structurally isomorphic with the externalized Logos itself. Then this idea 1 must contain, among many other things, a representation of itself: call this *idea 2*. This idea must then again contain an idea of itself, and so on. The conclusion is that the Logos is *quantitatively infinite* in the modern sense of the word. Thus there is an intimate connection between perfect self-reflection and infinity in the mathematical sense of the word.

Philo did not articulate, let alone pursue, this line of thought. He therefore lacked a clear understanding of the relationship between self-reflection on the one hand, and quantitative infinity on the other hand. But—or so I suggest—he was not far from grasping this connection.

25.6 Two Faces of Infinity

In the nineteenth century, in the work of Dedekind, several strands came together. It became clear that there are two ways to define quantitative infinity: two conceptions of quantitative infinity, if you like. On the one hand, one can say that a multitude is quantitatively infinite if the natural numbers—or a *simply infinite system*, in Dedekind's terminology—can be embedded into it. We might call this the *ordering conception* of infinity. This conception of infinity of course goes back at least to Aristotle. In his descriptions of potential infinity, he describes how such infinite orderings are "generated." On the other hand, Dedekind conceives infinity as a self-reflection property: a multiplicity is infinite if and only if there is a one-to-one onto correspondence between the multitude and a proper sub-multitude of itself (Dedekind, 1888, p. 64). This may be called the *reflection conception* of infinity. These two notions of infinity correspond to the now familiar distinction between an ordinal concept of number and a cardinal concept of number.

5 There may be a connection with a mystic idea according to which in the highest form of thought, the duality of subject and object of thought somehow dissolves.

Moreover, Dedekind knew that (in the presence of the Axiom of Choice), a multitude is ordering-infinite if and only if it is reflection-infinite.[6]

Dedekind believed that he could *prove* that there is an order-infinite collection (Dedekind, 1888, p. 66):[7]

> My own realm of thoughts, i.e., the totality *S* of all things which can be objects of my thought, is infinite. For if *s* signifies an element of *S*, then the thought s^0 that s can be an object of my thought, is itself an element of *S*. [...] then *S* is infinite, which was to be proved.

But Dedekind's argument has never been accepted as a real proof. Ultimately, the existence of an infinite collection is something that has to be *postulated*.

None of this in and of itself decides whether *actually infinite* collections exist. Clearly order-infinite collections of small transfinite order types can be conceived of in a potentialist way. But also Dedekind-infinity can be conceived of in a potentialist way, namely when the one-to-one correspondence between the whole and the part can be enumerated as a potentially infinite sequence.

If infinite pluralities of mathematical objects (numbers, or collections of numbers) are parts of the mind of God, then, because of the pure actuality of God, they must be actual infinities. To argue that certain infinite pluralities of mathematical objects are therefore also *bounded* and hence numerable (as Augustin did), would then be a further step, that could be taken, but could also be resisted. However, in the nineteenth century, speculation about the nature of mathematical infinity had mostly emancipated itself from theological speculation.[8] As far as I know, Dedekind did not appeal to medieval theology at any place in his theory of the foundations of mathematics. The only place where he uses a philosophical argument is his "proof" of the existence of infinite collections that was discussed above.

25.7 Cantor

Cantor was an exception to this emancipation process. He was a pre-eminent mathematician who did appeal to views and arguments from the history of philosophy and theology.

6 This follows from Dedekind, 1888, p. 72.

7 As Dedekind himself observed, a structurally similar "proof" of the existence of an order-infinite collection was given earlier by Bolzano. For a discussion of Bolzano's argument, see Tapp, 2019.

8 This process of emancipation began in the fourteenth century: see Biard and Celeyrette, 2005, Introduction.

We have already seen how Cantor agreed with Augustin that mathematical objects are ideas in the mind of God. Since the mathematical world is infinite, it must then be *actually* infinite in God's mind. Furthermore, Cantor followed philosophers-theologians like Philo and Augustine in taking it to be a fundamental principle that God is in a very strong sense epistemically transcendent (Cantor, 1962, *Abhandlungen zur Mengenlehre III*, Endnote to Section 4, p. 205, my translation):

> The Absolute can only be acknowledged, but never known, nor even approximately known.

In this famous quote Cantor takes us to already have "approximate knowledge" of God if we can "take the measure" of a dimension or compartment of God's mind. In particular, we would have approximate knowledge of God's mind if we could "measure" the extent of a dimension of it by means of the natural numbers, since the concept of the natural numbers is, for Cantor, perfectly clear. This leads him to posit what philosophers of mathematics take to be the first use of a *reflection argument in mathematics* (Cantor, 1962, *Abhandlungen zur Mengenlehre III*, Endnote to Section 4, p. 205, my translation)

> Whereas hereto, the infinity of the first number class (I) [i.e., the class of finite cardinal numbers] alone has served as such a symbol [of the Absolute], for me, precisely because I regarded that infinity as a tangible or comprehensible idea, it appeared as an utterly vanishing nothing in comparison with the absolutely infinite sequence of numbers.

The reasoning in this passage goes along the following lines. Suppose there is a one-to-one correspondence between the natural numbers and the mathematical world as a whole. Then a "measure has been taken" of the mind of God using a perfectly clear measuring stick (the natural numbers). So by elementary knowledge of the natural numbers we have knowledge of the mathematical part of the mind of God. But this is incompatible with the epistemic transcendence of God. Therefore the collection of the natural numbers must be of bounded size in comparison to the immeasurability of the mind of God. Bounds are given by numbers. Therefore there must be a number that measures the size of the natural numbers. This will then be a transfinite number: a bounded completed infinity.

There seems no obstacle to the *human knowability* of the number that measures the size of the natural numbers, since the knowledge of this number would give us no knowledge of the mathematical compartment of the mind of God, which immeasurably transcends the collection of the natural numbers. All this also holds for other "clear" collections of numbers, such as the rational numbers and the real numbers. So we might as well *try* to come to know the cardinal number of the natural numbers, as well as the cardinal numbers of other infinite collections, and how to calculate with these transfinite numbers. And this is of course exactly what Cantor did. In this way, he went beyond what Augustine thought possible: as we have seen, the latter's remarks were tentative, and he thought that in any event calculating with transfinite numbers is beyond the intellectual capacities of humans.

Cantor's reflection argument is restricted in scope. The Burali-Forti argument shows that the plurality of all ordinal numbers does *not* form a set. So, for Cantor, the infinity of all ordinals cannot be a "tangible, comprehensible idea," and is therefore not subject to a reflection principle.[9] This is somewhat puzzling, though, since the definition of the concept of ordinal seems quite perspicuous.

25.8 Reflection in Set Theory

Cantor has been credited with being the first to make use of a set-theoretic reflection principle.[10] Actually, the philosophical motivation of modern set-theoretic reflection principles is closer to Philo's reasoning than to Cantor's reasoning about the Absolute. Set-theoretic reflection principles center around the concept of *indiscernibility*. They somehow express that the set-theoretic universe V is indistinguishable from certain proper parts of V. It is of course not completely clear which notion of indiscernibility Philo had in mind: perceptual indiscernibility, epistemic indistinguishability in general, semantic indiscernibility… In modern set theory, the focus is firmly on a form of *semantic* indiscernibility. Like in Philo's reflection from God to certain angels, no distinction is made in modern set-theoretic reflection between "clear" and "unclear" or "indefinite" infinities.

The general idea between set-theoretic reflection principles is the following: a mathematical statement or collection of statements is true of the universe V as a whole if and only if it is true of certain parts of V when all quantifiers are restricted to these parts of V. The notion of reflection in this form "is probably the most universally accepted rule of thumb in higher set theory" (Maddy, 1988, p. 503).

This train of thought finds a clear expression in what is called the principle of *Montague-Levy reflection*. If we denote the α-th rank of the set-theoretic universe V as V_α, and the relativization of the quantifiers of a formula ϕ to V_α as $\phi^{V\alpha}$, then this schematic principle can be expressed as follows:

$$\phi \rightarrow \exists\alpha{:}\ \phi^{V\alpha} \qquad \text{(ML)}$$

9 In his later work, Cantor calls the plurality of all ordinals an *inconsistent multiplicity*.
10 See Hallett, 1984, pp. 116–118.

Montague and Levy showed that this principle is provable in standard set theory (Levy, 1960; Montague, 1961):

Theorem 1. *ZFC* $\vdash \phi \rightarrow \exists\alpha\colon \phi^{V\alpha}$.

Nonetheless, Montague-Levy reflection has hidden strength. Over ZFC *minus* the axiom of infinity and the axiom of replacement (call this theory ZC^-), ML is equivalent to the remainder of the axioms of ZFC:

Theorem 2. $ZC^- \vdash ML \leftrightarrow$ *Infinity* + *Replacement.*[11]

This shows that already ML is much more powerful than Cantorian reflection: a minimal "standard" model of ZC^- is V_ω, whereas a minimal standard model of ZFC is V_κ, where κ is the smallest strongly inaccessible cardinal.

Second-order (i.e., class-theoretic) analogues of ML can straightforwardly be formulated. Unlike Montague-Levy reflection, they are independent from ZFC, and are seen as plausible axioms of infinity. They are classed among the so-called "small large cardinal principles."

Principles of infinity that are independent of ZFC are called *large cardinal axioms*.[12] Second-order versions of ML are among the most modest such principles. Another modest large cardinal principle is the Axiom of Inaccessible Cardinals, which can be formulated as follows (Kanamori, 1994, Theorem 1.3):

Axiom 1 (IC). *There is a cardinal κ such that* $(V_K, \in) \models ZFC$.

This Axiom is significantly stronger than ML in the following sense: rather than postulating that every true sentence is reflected in some rank (possibly different ranks for different true sentences), IC claims that there is *one single* rank in which infinitely many true sentences (all theorems of ZFC) are reflected.

The vast majority of large cardinal principles can be expressed as *embedding principles*, which can be described as follows. An *inner model M* of *V* is a model of ZFC that contains all ordinals. The following is a prototypical embedding axiom:

Axiom 2 (MC). *There is a transitive inner model M and a nontrivial bijective class function j such that for all formulas* $\Phi(x_1,\ldots,x_n) \in L_{ZFC}$, *we have:*
$V \models \Phi(x_1,\ldots,x_n) \Leftrightarrow M \models \Phi(j(x_1),\ldots,j(x_n))$.

The first ordinal κ that is moved by j, which is called the *critical point* of j, is a cardinal number: it is called a *Measurable Cardinal*. Measurable cardinals are beyond what can be proved to exist by reflection principles that are variations on ML.

11 The Axiom of Choice play no role in the proof of this equivalence.

12 For an introduction to the theory of large cardinals, see Kanamori, 1994, Chapter 1.

Much larger large cardinals even than measurable cardinals can be postulated to exist by playing around with the conditions on j and by varying the notion of elementarity in Axiom MC.

Many set theorists take all embedding principles to be reflection principles. The reasoning behind this is straightforward. Inner models are proper *parts* of V. So MC postulates that V is ontologically reflected into itself as a part M of itself.

Since it contains all ordinals, M is a proper class. Therefore embedding principles do not postulate that V is reflected in a *relatively small* part of itself. The strength of embedding principles tends to be positively correlated to the degree of similarity of M to V itself.

The fact that embedding principles embed V into a large part of itself may be taken as a reason not to take embedding principles to be reflection principles in a strong sense of the word. But this defect of embedding principles can be overcome. Philip Welch has proposed the following large cardinal principle, which is called the set-theoretic *global reflection principle* (Welch and Horsten, 2016):

Axiom 3 (GRP). *There is an initial segment of the universe* V_κ, *and a nontrivial elementary embedding*

$$j: (V_\kappa, \in, V_{\kappa+1}) \rightarrow_e (V, \in, C)$$

with critical point κ, *and where* e *is*, as in MC above, *an elementary equivalence relation.*

If in this Axiom e is taken to be *second-order* elementary equivalence, then GRP is as strong as the embedding principle that is called the Axiom of *extendible cardinals*. It is worth noting at this stage that j plays an essential role in embedding principles such as MC and GRP: merely postulating elementary equivalence between V and an inner model (or a set) does not yield significant large cardinal strength.

The Axiom GRP postulates that V, together with all its proper classes, is reflected in a *set-sized*, and therefore "small" part (V_κ) of the universe, together with all of its sub-sets ($V_{\kappa+1}$). Thus GRP is a reflection principle in a strong sense of the word.

Most large cardinal axioms can naturally be and usefully are formulated as elementary embedding axioms. Since elementary embedding principles are class theoretic statements, this is an instance of the usefulness of second-order principles in set theory. By imposing additional conditions on j and on M, even stronger embedding principles are obtained. In particular, in stronger embedding principles, M looks more and more like V itself. Reinhardt observed that natural ultimate limit of this process therefore is to postulate a non-trivial embedding from V into *itself:*

Axiom 4 (R). *There is a non-trivial elementary embedding from V into V.*

But Axiom (R) was found to be incompatible with ZFC (Kunen, 1971):

Theorem 3. *ZFC* $\vdash$ *"There is no non-trivial elementary embedding from V into V."*

On the one hand, this phenomenon may be viewed as a *bad company problem* for set-theoretic embedding principles in general. The fact that a natural strong embedding principle is inconsistent might give us pause: it might lead one to doubt that certain restrictions of it are sound, or even consistent.

On the other hand, it has been observed that the proof of Theorem 3 makes *essential* use of the Axiom of Choice. Indeed, a rich structure theory of "choiceless cardinals" is currently being developed in the context of ZF (*without* the Axiom of Choice), where a choiceless cardinal is a cardinal that cannot exist if the Axiom of Choice is true. In particular, various strengthenings of Axiom R have been proposed, and in the absence of the Axiom of Choice, they appear (so far!) to be consistent. This gives the appearance of there being a whole *realm* of cardinals beyond the "choicy" cardinals. Some interpret this as indicating that the Axiom of Choice holds only up to a high level of the rank hierarchy, and that it has only has restricted validity beyond that.

25.9 Mathematical Warrant

It took a remarkably long time before it became clear that ontological reflection is entangled with quantitative infinity in the modern sense of the word. Since quantitative infinity lies within the compass of mathematics, ontological reflection has become of mathematical relevance. In particular, it has become of central importance in set theory, which can be seen as the mathematical investigation of quantitative infinity.

On the one hand, reflection phenomena have become part of set-theoretic experience. Montague-Levy reflection is only a tiny aspect of this: it is no exaggeration to say that the working set theorist encounters ontological reflection phenomena on an almost daily basis. On the other hand, strong reflection principles are *postulated* in set theory. This is not to say that strong reflection principles are universally or even very widely accepted as true in the set-theoretic community.[13] But they are highly trusted, in the sense that most set theorists regard it as unlikely that contradictions can be derived even from fairly strong set-theoretic reflection principles.

13 On this issue I wish to remain neutral.

There is a temptation to explain this trust of the set-theoretic community in reflection principles by appealing to philosophical arguments for their truth. The thought is that the truth of set-theoretic reflection principles follows from philosophico-theological arguments of the kinds that we have reviewed, *except* that in the arguments the concept of God is uniformly replaced by the concept of the set-theoretic universe (V).

The epistemic warrant for reflection principles is taken to be similar in nature to our epistemic warrant for basic principles of set theory in general. The basic principles of set theory have to be justified. But, since from a mathematical point of view the axioms of set theory are basic, their justification will have to involve non-mathematical notions. The justification of basic principles of set theory would thus be *partly* non-mathematical in nature. Let us call this the *justificatory account*.

The justification attempt for the basic axioms of set theory that has enjoyed most widespread support in the philosophy of mathematics is the *iterative conception of sets*. On this conception, the sets are generated in (cumulative) stages. At every successor stage $\alpha + 1$, for every plurality P of sets that exists at stage α, a *set* of the P's is generated. At any limit stage, the union is taken of all the sets that have been generated at earlier stages.[14] It is widely held that most axioms of ZFC can thus be justified.

The iterative conception is taken to be in tension with the attitude of taking the existence of choiceless cardinals seriously. Most philosophers believe that the Axiom of Choice can be justified on the basis of the iterative conception of set, along the following lines:[15]

> Suppose some family F of mutually disjoint non-empty sets exists at some stage α. Then there is a stage $\beta < \alpha$ such that all sets $a \in$ F (and *their* elements) exist at stage β. Then also at stage β some x-es exist such that, for each $a \in$ F, exactly one element of a is one of them (and nothing else is one of them). At stage $\beta + 1 \leq \alpha$, these x-es form a choice set C for F.

If this is true, then choiceless cardinals cannot exist in the same sense as the familiar choicy ones do. If we take the Axiom of Choice as a non-negotiable principle of set theory, then much of the current research into choiceless cardinals must be reinterpreted as being about inner models of ZF that *think* (1) that the Axiom of Choice does not hold unrestrictedly and (2) that there are choiceless cardinals.

14 A limit stage λ is thus the first stage at which all the sets that have been generated previously exist.

15 See, for instance, Schoenfield, 1977, p. 335. Boolos is an exception: he defends the iterative conception, but argues that the standard argument for the Axiom of Choice from the iterative conception of sets is *circular* (Boolos, 1971).

However, it is not clear that what I have called the justificatory account is faithful to set-theoretic practice. Most working set theorists—I claim—have never even heard of the iterative conception of set. After all, it belongs to philosophy (of mathematics), not to mathematics. At the same time, a growing number of set theorists behave as if choiceless cardinals are "just there" in V—but very high up the rank hierarchy. Indeed, because of this tension with set-theoretic practice, the justificatory account of basic set-theoretic principles seems to me a fundamentally wrong picture of mathematical epistemology, as I shall now all too briefly argue.

As mentioned above, if the justificatory account is true, then our fundamental warrant for basic set-theoretic principles—including those reflection principles that qualify as such—is partly non-mathematical in nature. The iterative conception, for instance, makes use of the notion of *generation in stages*, which is a non-mathematical concept. But, as Maddy[16] and others have emphasized, mathematics in general, and set theory in particular, is not in need of extramathematical epistemic support. All that matters for set theory to be in good epistemic standing, is that the beliefs and practices of set theorists are rational responses to the set-theoretic challenges that they are faced with.

Proof certainly plays a central role in set-theoretic practice, and proof is a rational road to belief. But adopting certain principles as basic can also be a rational act. In particular, set-theoretic reflection principles play a central *organizing role* in our set-theoretic practice, and this is one factor that explains why some of them are candidates for being basic axioms. Set theorists do not have to take this as their *explicit reason* for being (mathematically!?) warranted in adopting certain reflection principles as basic beliefs; nor do they have to be able to produce an *Inference to the Best Explanation* argument in order to support their belief in certain reflection principles. (After all, "best explanation" is also a non-mathematical notion.) All that matters, is whether their adoption of beliefs is a rational epistemic response: sometimes this is a matter of reasons (when a set theorist proves a theorem); at other times, it is a matter of *mathematical entitlement* that is not a matter of articulated motivating reasons.

25.10 Philosophical Warrant

What, then, becomes of the philosophical arguments that have been adduced to support the adoption of mathematical principles? Some of them may, for all that I have said in the previous section, still constitute *good* reasons for believing cer-

16 See Maddy, 2009.

tain mathematical principles. *If* they are good reasons, then everyone—philosophers and mathematicians alike—can make epistemic use of them. In this sense, philosophical reflection arguments would then still be epistemically valuable. It is just that set theory, in its current state of development, is in good epistemic standing even if it does not mention them.

The root of philosophical arguments for reflection principles lies in the thesis of the unknowability (and existence) of God. This thesis is supported by the Bible, and has long been widely accepted in Western philosophy. Augustine argued that the mathematical world forms a part of the mind of God, and on the basis of this made an unknowability claim concerning the mathematical universe. Currently, the thesis of the unknowability of the mathematical universe as a whole enjoys wide support even in the absence of the identification of the mathematical universe with a part of the mind of God.

The thesis of the unknowability of God and the thesis of the unknowability of the mathematical part of the mind of God are negative theses. Philo saw that such negative claims can be given a positive interpretation: it gives rise to *indistinguishability* claims (Welch and Horsten, 2016, pp. 95–96). If we identify the mathematical Absolutely Infinite with the set-theoretic universe V, then V is unknowable in the sense that we cannot single it out or pin it down by means of any of our assertions: no true assertion about V can be made that excludes other unintended interpretations that make the assertion true. In particular—and this is stronger than the previous sentence—no assertion that we make about V can ensure that we are talking about the mathematical universe rather than an object in this universe. So if we do make a true assertion ϕ about V, then there exist sets s such that ϕ is also true when it is interpreted over s. Moreover, this should hold not only for individual claims about V, but also about infinite collections of claims about V. Gödel held the view that all sound large cardinal principles are somehow reducible to reflection principles (Welch and Horsten, 2016, Section 8.7.9).

The paradox that Philo already identified remains: how can the indistinguishability claim even unambiguously be *stated* if no expression in our language singles out God (or V)? In modern set theory, this is dealt with by restricting the language for which the indistinguishability claim is supposed to hold: the official language of set theory does not contain an individual expression that refers to the universe as a whole.

If philosophical reflection arguments have justifying force, then *which* reflection principles do they justify? The philosophical reflection principles that we have considered aim at supporting statements that express ontological reflection of an absolutely infinite realm into a humanly intelligible entity. For set theory, the humanly intelligible objects are the sets. Therefore, Montague-Levy reflection captures the conclusion of the philosophical reflection arguments that we have discussed in a very weak sense, and variants of the Global Reflection Principle captures it in a much stronger sense. Standard embedding formulations of large cardinal axioms posit the existence of an embedding of V into an inner model. Inner models contain all the ordinals, and are therefore absolutely infinite. It is thus not at all clear to what extent inner models can be considered as humanly intelligible objects in the same sense as sets are. Hence it is not clear that standard embedding axioms can be directly justified by philosophical reflection arguments of the kind that we have considered. In the light of this, let me therefore end this article with a:

Question. *Can certain embedding principles that are (outright or in consistency strength) significantly stronger than embedding principles that posit extendible cardinals be formulated as or approximated by principles that state that certain rank initial segments of V are significantly indistinguishable from V as a whole?*

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