NOTE ON AN OBJECTION OF LIFSCHITZ AGAINST SHAPIRO'S "EPISTEMIC ARITHMETIC"*

Leon Horsten

Abstract In his paper "Calculable Natural Numbers", Vladimir Lifschitz describes what he takes to be "undesirable consequences" of Stewart Shapiro's theory of epistemic arithmetic (p.186-187).

Although it is not immediately obvious from the allusive passage in which he formulates his misgivings about Shapiro's system *why* he thinks that the consequences which he mentions are objectionable, it is clear that his objection pertains to Shapiro's regimentation of statements containing classical as well as constructive existential quantifiers (let us call these statements of *relative constructibility*).

In our presentation, we investigate in some detail the notion of relative constructibility on which Lifschitz's objection is based. In particular, we address the following questions:

1. Does Shapiro's theory of epistemic arithmetic entail false sentences about the logical properties of this notion of relative constructibility?
2. Is the language of Shapiro's system of epistemic arithmetic strong enough to adequately express this notion of relative constructibility?

Our answer to the first question is a qualified no, and our answer to the second question is a qualified yes.

In the remainder of our paper we show that the notion of relative constructibility (as we construct it) gives rise to a *hierarchy* of statements of relative constructibility, which in our opinion merits further investigation.

Shapiro has proposed a formal theory in which both constructive and non-constructive aspects of arithmetic can be expressed. The purpose of his theory is to "integrate" classical and intuitionistic arithmetic. He intends to accomplish this by adding an epistemic sentential operator (K) to the formal language of

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first-order arithmetic. S4 deduction rules are formulated for K, and the theory that results from adding these rules to the axioms of first-order Peano arithmetic is called "Epistemic Arithmetic" (EA). The epistemic operator of EA is to be interpreted as "It is ideally, or potentially knowable that" [5, page 25].

Shapiro proposes the following translation V_0 of the *connectives* of the language of Heyting arithmetic (HA) to the language of EA (where we indicate by means of a subscript i ("intuitionistic") that a connective belongs to the language of HA) [5, page 25]:

$$V_0(A \wedge_i B) = KA \wedge KB$$

$$V_0(A \vee_i B) = KA \vee KB$$

$$V_0(A \rightarrow_i B) = K(KA \rightarrow KB)$$

$$V_0(A \leftrightarrow_i B) = K(KA \leftrightarrow KB)$$

$$V_0(\neg_i A) = K\neg KA$$

$$V_0(\forall_i xA) = K\forall xA$$

$$V_0(\exists_i xA) = \exists xKA$$

Shapiro then constructs a translation V from *formulas* of the language of HA to the language of EA [5, page 25]. If we agree to indicate by means of a subscript i that a formula belongs to the language of HA, then we can paraphrase this translation V as follows:

- for atomic formulas:

$$V(A_i) = KA_i$$

- for complex formulas:

$$V(A \wedge B)_i = K(V(A_i)) \wedge K(V(B_i))$$

$$V(A \vee B)_i = K(V(A_i)) \vee K(V(B_i))$$

$$V(A \rightarrow B)_i = K(K(V(A_i)) \rightarrow K(V(B_i)))$$

$$V(A \leftrightarrow B)_i = K(K(V(A_i)) \leftrightarrow K(V(B_i)))$$

$$V(\neg A)_i = K\neg K(V(A_i))$$

$$V(\forall xA)_i = K\forall xV(A_i)$$

$$V(\exists xA)_i = \exists xKV(A_i)$$

Nicolas Goodman has shown that this translation V is faithful [1]:

Theorem 1 *For every formula A_i of the language of HA :*

$$\vdash_{HA} A_i \quad \Leftrightarrow \quad \vdash_{EA} V(A_i).$$

This theorem does not guarantee that the clauses of the definition of V give the meaning of the intuitionistic sentences.[1] It does entail that to the extent that HA adequately axiomatizes intuitionistic arithmetic, one cannot challenge the thesis that V is meaning-preserving on the basis that if we take the formulas in the range of V to have the intuitionistic meaning that V suggests, then intuitionistic formulas which should be provable aren't, or intuitionistic formulas which shouldn't be provable are.

Let us for a moment accept that the clauses of V_0 give the meaning of the intuitionistic connectives, and that the clauses of V give the meaning of the intuitionistic formulas. Then not only formulas of classical arithmetic and of intuitionistic arithmetic are provable in EA, but also formulas "of mixed constructivity", i.e. formulas which contain translations of intuitionistic connectives which are not equivalent to intuitionistic formulas or to classical formulas. Presumably we have some intuitions about which formulas of mixed constructivity should be provable and which shouldn't. Then we can still attempt to refute the thesis that V_0 and V are meaning-preserving in the above sense on the basis that some formulas of mixed constructivity which should be provable aren't, or that some such formulas which shouldn't be provable are. This seems to be what Lifschitz intends to do when he cryptically formulates his argument against Shapiro's translation [4, pages 185–187]:

> "One might think then that assertions about natural numbers involving both classical and constructive quantifiers can be expressed in Heyting arithmetic by using $\neg\neg\exists$ for the former and $\exists$ for the latter. This method seems, however, unsatisfactory. Take an assertion of the form: "there exists (non-constructively) an x such that one can calculate a y such that $A(x, y)$". The translation
>
> $$\neg\neg\exists x \exists y A(x, y)$$
>
> suggested above does not work: it makes *both* quantifiers non-constructive, not only the first one. This difficulty can be seen even more clearly if we consider a slightly more complicated assertion: "there exists (non-constructively) an x such that, for every w, one can calculate a y such that $A(x, w, y)$". The translation
>
> $$\neg\neg\exists x \forall w \exists y A(x, w, y)$$

[1] For arguments against the thesis that V is meaning-preserving in this sense, see [2] and [8]. For reasons we cannot go into here, we do not find their arguments convincing. In the presence of a faithfulness theorem, as is the case for V, one of the few plausible ways of arguing for (against) the thesis that a given translation is meaning-preserving seems to be to argue convincingly to the effect that the translation is (not) more or less implicitly used by ordinary language users. If this is so, then the fact that the clauses for V approach Heyting's proof conditions for the intuitionistic connectives provides some support for the thesis that V is almost meaning-preserving (if the connectives of the language in which Heyting stated his proof conditions are classical).

does not work; neither, it seems, does anything else,...

...It seems, furthermore, that the approach of Shapiro and Myhill is subject to a similar criticism to the one brought forward at the beginning of this section in connection with the idea of using $\neg\neg\exists$ for classical existence. In fact, their constructive existential quantifier $\exists_i$ is defined by

$$\exists_i x A \equiv_{def} \exists x K A$$

where K is the new operator. The theorems

$$KA \rightarrow A$$

$$KA \rightarrow KKA$$

$$\exists x KA \rightarrow K\exists x A$$

imply

$$\exists x \exists y KA \leftrightarrow \exists x K \exists y KA$$

i.e.,

$$\exists x \exists_i y A \leftrightarrow \exists_i x \exists_i y A$$

The obvious equivalence

$$\exists x \exists_i y A \leftrightarrow \exists_i y \exists_i x A$$

also seems undesirable."

Let us concentrate on $\vdash_{EA} \exists x \exists_i y A \leftrightarrow \exists_i x \exists_i y A$.[2] The right-to-left direction of this formula seems unproblematical, so let us consider the left-to-right direction $\exists x \exists_i y A \rightarrow \exists_i x \exists_i y A$. Lifschitz does not tell us *why* he finds this implication objectionable. But presumably he objects to this because it seems to say that if there is a number x for which we can effectively find a number y such that $A(x, y)$, then that number x can be effectively found too.

But the formula $\exists x \exists_i y A \rightarrow \exists_i x \exists_i y A$ does not say that (even though this reading is strongly suggested by the absence of a subscript i in the first quantifier). The first existential quantifier of its antecedent is also constructive. This can be seen by reformulating the antecedent of this formula as $\exists x \exists y KA$, which says that there exist numbers x and y which (both!) can be known to stand in a relation A to each other. In any case, it is important to note that the temptation to read the formula in the incorrect way described above is due exclusively to V_0. So if we dispense with V_0 altogether, while retaining V [3], then nobody will be tempted to read such formulas in an incorrect way.

[2] Our explanation and assessment of the *prima facie* counterintuitivity of the formula $\exists x \exists_i y A \leftrightarrow \exists_i x \exists_i y A$ can be adapted in a straightforward way to the *prima facie* counterintuitivity of the formula $\exists x \exists_i y A \leftrightarrow \exists_i y \exists_i x A$. Therefore this latter formula will be given no further attention in this paper.

[3] This is done, by the way, in all papers on EA other than Shapiro's that we are aware of.

However, one question remains. It is easy to formalize in the language of EA that "we can effectively find an x such that there is a y (which perhaps cannot be effectively found) such that $A(x,y)$": this can be expressed as $\exists x K \exists y A(x,y)$. But how *can* we express in the language of EA that "there is an x (which perhaps cannot be effectively found) such that we can effectively find a y such that $A(x,y)$"? Our discussion so far of Lifschitz's objection shows that $\exists x \exists y K A(x,y)$ will not do. If it turns out that the mathematical content of this statement cannot be expressed in EA, then we have hit upon a serious limitation of the expressive power of this system.

Before we attempt to tackle this problem, observe that it is not immediately clear what the expression "there is an x (which perhaps cannot be effectively found) such that we can effectively find a y such that $A(x,y)$" is intended to mean.

First of all, for the statement to be interesting, it must be intended to express that for some x we can effectively find a y which can be shown to stand in the relation A to x. For we can effectively find *any* natural number; it is showing *of* a natural number *that* it has some property that can be nontrivial! And this brings us to a point about Lifschitz's own epistemic theory of arithmetic, his theory of "Calculable Natural Numbers". Unlike Shapiro, he uses an epistemic *predicate* $K(x)$, which should be read as "x is calculable" ([4, page 174]). But by what we have just said, it follows that constructive or effective existence should not be considered to be a *property* of numbers; statements asserting constructive existence of a number with a certain property should be formalized using a de re modality, as expressing that it is provable *of* that number *that* it has the relevant property, and not that the number can be constructed *and* that it has the property, as is done in Lifschitz's system.[4]

Next, either the effective or constructive procedure for finding y takes the number x as an input or it doesn't.

In the latter case, the intended meaning (or something close to it) can most probably be expressed in the language of EA as $\exists x \exists y K A(x,y)$. If there is a number x such that we can effectively find a number y for which we can show that $A(x,y)$, then this number x can be effectively found too (and conversely). Shapiro suggested during the discussion of my paper at the conference that perhaps Lifschitz's reading of "there is an x (which perhaps cannot be effectively found) such that we can effectively find a y such that $A(x,y)$" can be expressed as $\exists y K \exists x A(x,y)$. I do not think that this is the reading which Lifschitz intended, but in any case, in this reading the value of x is also not taken as an input of the procedure for finding the value of y.

The former case is more interesting. If the effective procedure for finding y takes x as an input, then the statement intends to express the constructibility

[4] Shapiro suggests that we understand the meaning of Lifschitz's predicate $K(x)$ as being determined in part by the context in which it occurs ([7, page 7]). This solves the problem, but it should not be forgotten that this amounts to making implicit what could be expressed explicitly (as the system EA shows).

of a y such that $A(x,y)$ relative to a value of x such that $\exists y A(x,y)$. In such cases, when a number y can be effectively found when the value of a certain other number x, which stand in a certain relation to y, is given, we will speak of *dependent constructibility*, and we will say that y is constructible relative to a value of x.

Using a form of Church's thesis, we can express the "interesting" reading as

$$\exists x \exists y A(x,y) \wedge \exists e \forall x(\exists y A(x,y) \rightarrow \varphi_e(x) = y)$$

where $\varphi_e(x) = y$ expresses that the Turing machine with Gödel number e, when it is started on the number x, yields as output the number y. $\varphi_e(x) = y$ can be expressed in the language of EA using Kleene's T-predicate and the U-funcion symbol. But of course we want our statement to be expressed in a more direct manner, which does not depend on the identification of constructive procedures with recursive ones (this was precisely the point of introducing the epistemic operator!). The following formula of the language of EA does this:

$$\exists x \exists y A(x,y) \wedge K \forall x(K \exists y A(x,y) \rightarrow \exists y K A(x,y)) \tag{1}$$

The first conjunct ensures non-vacuousness, whereas the second conjunct (roughly) says that there is a proof which transforms any proof of an x that it stands in the relation A to some y into a proof that this y can also be effectively found.

We believe that (1) expresses the kind of relative constructibility which Lifschitz had in mind when he formulated his objection against EA. Moreover, examining a few statements expressing similar forms of relative constructibility, one sees that this way of expressing Lifschitz's dependent constructibility can be generalized to cover *all* such forms of dependent constructibility.[5] As an example, let us return to Lifschitz's expression "there exists (non-constructively) an x such that, for every w, one can calculate a y such that $A(x,w,y)$." [4, page 86]. The "interesting" reading of this statement can be formalized in the language of EA as:

$$\exists x \forall w \exists y A(x,w,y) \wedge K \forall x(K \forall w \exists y A(x,w,y) \rightarrow \forall w \exists y K A(x,y)) \tag{2}$$

The notion of dependent constructibility can be iterated, which gives us a means to express higher "degrees of constructibility". This gives rise to a *constructive hierarchy*, which we shall call C. C is the union of the sets of formulas C_0^0, C_1^0, C_2^0, ... C_0^1, C_1^1, C_2^1, ... of the language of EA, which are defined as follows. $C_0^0 = C_0^1 = \Delta_0$, the set of quantifier-free fomulas of the language of EA. For every number n, and every formula ϕ, if there

[5] An exception needs to be made for expressing some functional independences of choice of values in statements containing iterated dependent constructibility, such as "there is an x, relative to which we can effectively find a y, and there is a z, relative to which we can effectively find a u, such that $A(x,y,z,u)$". To express such statements we need branching quantifiers (or second-order quantifiers). But this is not surprising, since we also need branching quantifiers (or second-order quantifiers) to express such independences in classical predicate calculus.

is a formula $\phi^* \in C^0_n$ such that $\phi = \exists x K\phi^*$, then $\phi \in C^1_{n+1}$. For every number n, and every formula ϕ, if there is a formula $\phi^* \in C^0_n$ such that $\phi = \exists x\exists y\phi^* \wedge K\forall x(K\exists y\phi^* \rightarrow \exists yK\phi^*)$, then $\phi \in C^0_{n+1}$. We can then use Lifschitz's notation, this time in a non-misleading way, to abbreviate the formulas thus introduced in the hierarchy: we leave the C^0_0-formulas as they are; we abbreviate formulas of the form $\exists xK\phi^*$, with $\phi^* \in C^0_n$, as $\exists_i x\phi^*$; we abbreviate formulas of the form $\exists x\exists y\phi^* \wedge K\forall x(K\exists y\phi^* \rightarrow \exists yK\phi^*)$, with $\phi^* \in C^0_n$, as $\exists x\exists_i y\phi^*$. In this way we obtain formulas of the form $\exists_i x\exists y\exists_i z\ldots\phi$ (belonging to C^1_n for some n) and $\exists x\exists_i y\exists z\ldots\phi$ (belonging to C^0_n for some n), expressing ever more complex forms of dependent constructibility. To conclude, we close every C^0_n and C^1_n under EA- equivalent formulas, i.e., if $\phi^* \in C^0_n$ ($\phi^* \in C^1_n$), and ϕ and ϕ^* are provably equivalent in EA, then also $\phi \in C^0_n$ ($\phi \in C^1_n$).

The hierarchy which is thus defined is not exhaustive: most formulas of the form of (2), for instance, do not belong to C. It would be interesting to try to show that the hierarchy C is strict, and to extend the hierarchy to that it becomes exhaustive, but that goes beyond the scope of this paper.

It is now time to summarize our findings. Lifschitz's objection shows only that it is best to omit Shapiro's translation V_0. His objection does not tell against Shapiro's claim that the formulas in the range of V come fairly close to expressing intuitionistic propositions. Nor does his objection, as far as we can see, point to a limitation in the capability of EA to express forms of mixed constructivity. It does seem to us, however, that the structure of the complex forms of mixed constructivity which are expressible in EA merits further investigation.

Trying to show that there are formulas of mixed constructivity which are provable in EA even though they shouldn't be seems altogether an unpromising undertaking: the axioms of EA appear to be true of their intended interpretation and the rules of inference seem to be truth-preserving. Perhaps one should rather investigate whether there are principles of mixed constructivity which should be derivable but which aren't. Elsewhere (see [3]) we investigate the bearing of principles of mixed constructivity which are independent of EA on the acceptability of so-called "problematic" constructivistic principles, such as the intuitionistic version of Church's thesis and Markov's principle. For all we know, it may even be that there are intuitively valid logical principles of (some notion of) relative constructibility which are not provable in EA. But so far, we haven't managed to find one.

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Leon Horsten
Research Assistant of the
Belgian National Fund for Scientific Research
Center for Logic and Philosophy of Science
Institute for Philosophy
Catholic University of Leuven
Kardinaal Mercierplein 2
B-3000 LEUVEN