

Computational Structuralism[†]

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According to structuralism in philosophy of mathematics, arithmetic is about a single structure. First-order theories are satisfied by (nonstandard) models that do not instantiate this structure. Proponents of structuralism have put forward various accounts of how we succeed in fixing one single structure as the intended interpretation of our arithmetical language. We shall look at a proposal that involves Tennenbaum's theorem, which says that any model with addition and multiplication as recursive operations is isomorphic to the standard model of arithmetic. On this account, the intended models of arithmetic are the notation systems with recursive operations on them satisfying the Peano axioms.

[A]m Anfang [. . .] ist das Zeichen.
(Hilbert [1935], p. 163)

1. Structuralism and Nonstandard Models

Benacerraf [1965] laid the foundations for structuralism in the philosophy of mathematics; he claimed that arithmetic—and other mathematical theories—should not be conceived as topics with a specific and fixed subject matter; rather, arithmetic is about a certain *structure* irrespectively of the objects which form the domain of the structure. This is the structuralist credo in a nutshell.

On the structuralist account, arithmetic is about a single structure: *the* standard model of arithmetic.¹ All models instantiating this unique structure are intended.² All other models—whether

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¹ 'The standard model of arithmetic' is a misnomer for the structuralist. For him the standard model of arithmetic is a structure that is instantiated by various models, *e.g.*, by the finite von Neumann numbers and the Zermelo numbers.

² In this paper we take no stand on the ontology of structures. By using terms like 'instantiate' that derive from the theory of universals we do not commit ourselves to the view that

they are countable or not—do not instantiate the structure arithmetic is about.

Structuralism is threatened by a formal problem. First-order Peano arithmetic and all other sound first-order arithmetical theories have models that instantiate different structures. The nonstandard models of arithmetic are definitely *not* intended models. They exemplify structures different from the standard model.

Thus, on the one hand, structuralists deny that there is only one single model of arithmetic. This seems sensible because it does not matter from what material the standard model is built. Mathematicians will be content with any model isomorphic to, say, the von Neumann ordinals or the Zermelo numbers. On the other hand, not all models of first-order arithmetic are intended models. Models not isomorphic to the von Neumann ordinals need to be banned. Unfortunately no first-order axiom is capable of singling out the standard model.

Structuralists have tried several strategies to rule out nonstandard models of arithmetic. Benacerraf [1965] required that the structure is an ω -sequence *and* that the ordering on the elements is recursive. Of course, only the standard model has order-type ω . Thus the additional requirement that the ordering of the structure is recursive is not needed for ruling out standard models. Finally Benacerraf [1996] dropped the requirement of recursiveness and concluded that ‘any old ω -sequence would do after all’. Thus nonstandard models are simply ruled out by this restriction on the order-type of the model.

This requirement, however, is very bold and, in a sense, it begs the question. It is like requiring that the model should be isomorphic to the standard model of arithmetic. But it is exactly the task to spell out what the natural numbers are, and the structuralist has just dismissed the conception of the natural numbers as a single ‘given’ model. Using the concept of the natural numbers in explaining what the natural numbers are *is* begging the question. And the concept of an ω -sequence comes very close to doing just this. Benacerraf cannot explain the notion of an ω -sequence by saying that an ω -sequence is an ordering with the same order-type as the natural numbers. Usually the concept of an ω -sequence is defined in a set-theoretic framework: An ω -sequence is something isomorphic to the von Neumann ordinals. The von Neumann ordinals are defined in set theory. Now the structuralist has to look at the structure of sets. The problems of nonstandard models recurs here because there are also nonstandard models of set theory (if set theory is consistent). That is, there are models of set theory whose finite ordinals are not well-ordered ‘from outside’, but are

structures are actually universals. What we say might be compatible with an *ante rem* or an *in rebus* conception of structures or even with ‘nominalist’ structuralism where talk about structures is taken as a mere abbreviation for talking about isomorphic models.

well-ordered from the perspective of the model. Thus in order to make sure that the predicate ‘ ω -sequence’ applies only to well-orderings, one has to make sure that the model of set theory employed is standard. The view of a fixed given model for a theory has been already rejected by the structuralist. So the same problem as for the numbers is re-instantiated. Consequently most structuralists adopted other strategies for pinning down the standard models.

There is another obviously unacceptable method for ruling out non-standard models: One can require that every element in the model is named by some numeral of the language of arithmetic. Since nonstandard numbers are not named by a (standard) numeral, only the standard model meets this requirement. This method is not much better than Benacerraf’s direct ban on nonstandard models. It is not implicit in the first-order axioms of number theory that the numerals name all objects. For the first-order axioms do not say anything about naming of objects directly. Of course one can define (via Gödel coding) the relation that obtains between (codes of) closed terms of arithmetic and the named objects, but one can prove from the axioms of Peano arithmetic (and much weaker theories) that any object is named by its numeral. This means only that Peano arithmetic proves that for any object (whether it is standard or not) there is a numeral for it. In the case of a nonstandard element its code will be nonstandard as well. Only when we have understood what a standard numeral is can we rule out nonstandard models by requiring that every element of the model is named by some *standard* numeral. Thus this method begs the question: in order to apply it, we must be able to distinguish between standard and nonstandard numerals. And a model will be standard if it does not contain nonstandard numerals.

By far the most popular approach to ruling out nonstandard models relies on second-order logic. This approach has been advocated forcefully by Shapiro [1991], [1997]. But this approach is also problematic. Understanding second-order consequence presupposes a prior understanding of the notion of power set (at least over an infinite set). McGee [1997] proposed a refined version that relies on the extendability of the induction principle to new languages; this version has the advantage of requiring second-order logic only in a very concealed and seemingly less disquieting way. However, any kind of second-order approach will make use of the power set of the set of natural numbers. This power set, we submit, is far more problematic than the notion of the natural number itself. For the independence phenomena revealed by Gödel and Cohen suggest that the notion of the power set of the natural numbers may be inherently indeterminate or essentially relative.

In this paper, however, we will not go into the details of structuralism based on second-order logic. Instead we will return to Benacerraf’s original attempt to fix the standard model up to isomorphism. He stated in his

[1965] that any recursive ω -sequence could serve as an intended model for arithmetic. As mentioned above, he later discarded the ‘recursive’ part of this requirement. The remaining postulate that the model be an ω -sequence is surely sufficient, but, as we have seen, it begs the question. In this paper, we would like to investigate the prospects of a structuralist approach that drops the ‘ ω -sequence’ part of Benacerraf’s original requirement but retains the ‘recursive’ part.

2. Tennenbaum’s Theorem

Benacerraf ([1965], p. 275–277) has argued that it should be possible to determine effectively which one of two given elements of a model is greater. Here we do not go into the details of Benacerraf’s account, but this requirement has at least some initial plausibility: it should be possible to compute which one of two given numbers is greater. A model where there is no general procedure for finding out which of two objects of its domain is the greater should not be admitted as a model of arithmetic.

We shall extend this requirement. We agree with Benacerraf that the relation of being greater should be computable in the model. If the ordering of the objects ought to be recursive, then also the operations of addition and multiplication ought to be recursive: As there ought to be an algorithm for deciding the ordering of the elements of the model, there ought to be also algorithms for addition and multiplication. If the restriction to models with decidable $<$ -relation is sensible, then plus and times must be decidable as well. For it seems also of fundamental importance that we can compute the sum and the product of any two given numbers. If we cannot even in principle determine the sum and product of any two given numbers, then the objects in question are not numbers. Numbers are something we can calculate with; if we cannot calculate with objects, then they are not numbers.

So we impose the condition that the sum and the product of two given numbers can be computed. Actually it would have been sufficient to insist on the computability of the sum. We are also committed to the computability of many further operations like exponentiation on the natural numbers. But since their computability is not needed for ruling out nonstandard models, there is no point in specifying exactly which operations ought to be computable. Therefore we impose the following restriction on models:³

REC1: In an intended model the relation $<$ and the operations of addition and multiplication are recursive.

REC1 suffices for ruling out nonstandard models. Tennenbaum [1959] proved that only the standard model of Peano arithmetic satisfies REC1, that is, the operations of addition and multiplication are only recursive in the standard model.

³ See Horsten [2001], pp. 126–127.

Now Benacerraf's original restriction to ω -sequences is not needed anymore. We submit that it would have been more sensible to discard the requirement that the model is an ω -sequence than the requirement of recursiveness. The recursiveness requirement is as effective in ruling out nonstandard models as the bold restriction to ω -sequences. Benacerraf's original requirement to the effect that the ordering relation on the model is recursive is not sufficient for banning nonstandard models. For there are models of arithmetic where the $<$ -ordering is decidable, while addition and multiplication are not (Kaye [1991], p. 157, Exercise 11.10).⁴

3. Codings

The account presented so far—and Benacerraf's account [1965]—suffer from one blunt mistake: Recursiveness is defined only for predicates and functions *on natural numbers*. Restriction REC1, as presented above, applies only to models with domains that are subsets of the set of natural numbers. For there is no general notion of recursiveness that applies to arbitrary relations and functions on arbitrary objects whether they are mathematical, concrete, or whatever. What does it then mean that the ordering $<$ or the operation of addition on these elements is computable? For instance, is the natural ordering of the von Neumann ordinals recursive? We would like to answer this question in the affirmative, but recursion theory does not tell us whether this relation is recursive.

The obvious move for the mathematician is *coding*. Given a model of arithmetic one can try to code the elements of the domain of the model by natural numbers and then check whether the ordering, addition, and multiplication as induced by the coding relation are recursive on these codes. This is how Tennebaum's theorem is generally formulated: If a model can be coded in the natural numbers so that the induced operation of addition is recursive, then the original model is isomorphic to the standard model (see Kaye [1991], p. 153).

The coding relation need not be 'effective' in any sense. Effectiveness is subject to similar conceptual difficulties as recursiveness itself. The coding of set of expressions can be effective or not, but in general there is no fixed notion of effectiveness.⁵

For the purpose of this paper a coding is simply a one-one-mapping of the class of the objects in question to the set of the natural numbers. Thus REC1 may be rephrased as follows:

REC2: For every intended model there is a coding of the set of its elements such that the relation $<$ and the operations of addition and

⁴ However, the recursiveness of addition alone is sufficient to guarantee the standardness of the model. So REC1 and its variants with the requirement on multiplication deleted would serve the purpose.

⁵ In a later section we will qualify this judgement.

multiplication on the codes, as they are induced by the relations on the intended model, are *recursive*.

Obviously there is no such coding of an uncountable model of arithmetic, and consequently such models are trivially ruled out by REC2 as intended models.

REC2 also rules out all models where the field of $<$, together with the ordering relation $<$, does not constitute an ω -sequence. If $<$ is not of order-type ω , there will be nonstandard elements, and the codes of the elements of the model together with the respective interpretation of the relation and function symbols constitute a nonstandard model of arithmetic. By Tennenbaum's theorem, addition cannot be recursive. Therefore the model is ruled out as an intended model. In other words, REC2 determines the intended model up to isomorphism, as is required by structuralism concerning the natural numbers.

4. Recursiveness

One can mount a sceptical challenge: how *do* we know that the operations of addition and multiplication are recursive?⁶ This worry can be seen as a variation on the sceptical scenario that was investigated by Kripke in his *Wittgenstein on rules and private language*.⁷ An immediate reply would be that if these relations were not recursive, then we could never have learned the rule for using them. But, of course, Wittgenstein and Kripke have taught us that the debate does not end here: it is by no means a simple matter to say with confidence *what* we have learned when we have learned to add natural numbers. In this paper, we will leave the matter here, for we have nothing to add to the extensive debate which has followed the publication of Kripke's book on Wittgenstein.

There is an obvious objection to REC2 that we do have to answer. We have rejected Benacerraf's postulate that the model be an ω -sequence because the notion of an ω -sequence should not be presupposed in an account of what the numbers are. REC2 does not seem to fare better in this respect. On the usual account, recursiveness is defined for functions and sets of natural numbers. Thus the notion of a natural number precedes all notions of recursion theory. Thus, it seems, the structuralist who relies on REC2 in order to fix the standard model is subjected to the criticism that he presupposes the natural numbers in order to explain what the natural numbers are.

Dean [2002] defends the outlined application of Tennenbaum's theorem by conceiving recursiveness (or decidability) as a basic notion that is not preceded by the notion of number, function *etc.*: recursiveness is claimed

⁶ This question was raised independently, in conversation, by John Burgess and Paul Benacerraf.

⁷ See Wittgenstein [1958], Kripke [1982].

to be more basic and understandable independently from more advanced mathematical concepts. In fact, the notion of an *effective procedure* and thus of computability in the informal sense does not presuppose number theory or even set theory. Effective procedures do not only apply to numbers, but also to other objects. For instance, there is a procedure for checking whether a given expression is a well formed formula or not. The notion of an effective procedure applies also to methods for determining the species of a mushroom or the determining whether a ball bearing is still in order. Of course a lot can be said about following the rules of a procedure and about the ontology of procedures in general.

This *practical* notion of computability is distinguished from the *theoretical* notion of computability (and recursiveness). The theoretical notion of recursiveness is a purely mathematical notion; it applies primarily to numbers (*viz.* functions and relations of numbers *etc.*) and one may think of this notion as defined in set theory. For instance, the set of μ -recursive functions is easily defined in set theory (or another mathematical framework). The practical notion, in contrast, is not defined in set theory and does not completely belong to theoretical mathematics. For mathematicians there is usually no need to distinguish between the notions as long as the notions of decidability, recursiveness, effective procedure, *etc.* are only applied to functions and relations on natural numbers.

Of course there is an intimate connection between the practical and the theoretical notion of recursiveness. In the proof of Tennenbaum's theorem one relies on the theoretical notion of recursiveness. In order to apply Tennenbaum's theorem for ruling out nonstandard models, we have to assume that a practically recursive operation is also recursive in the formal sense. That is, we are appealing to Church's thesis.

Although for mathematical purposes there is no need to distinguish between the practical and the theoretical notions if Church's thesis is assumed, there is a crucial difference between the notions that is exploited by the structuralist, when he tries to apply Tennenbaum's theorem for banning nonstandard models of arithmetic. The theoretical notion is obviously useless to the structuralist because it singles out a certain class of functions and relations in a way that presupposes the natural numbers or set theory. If recursiveness is understood as a more basic notion—as Dean suggests—then it must be the practical notion. Thus we shall look at this notion in some detail. In particular, we shall look at the ontological foundations of this sort of recursiveness.

5. Practical Decidability

Usually textbooks in recursion theory start with a presentation of the informal and practical concept of computability, and to this end the informal concept of an effective procedure or algorithm is employed.

Algorithms tell you how to manipulate symbols; they do not tell you what to do with numbers if the structuralist account of natural numbers is presupposed. For one cannot 'do' anything with a number on the structuralist account where numbers are not particular fixed objects with an internal structure.

For instance, given two natural numbers k and n , we say that there is an effective procedure for determining whether $k^2 = n$. But for the structuralist, n can be anything. n might be a particular set, such as the corresponding von Neumann ordinal or the Zermelo number, or some other object. Only its status among other objects makes it the number 2. No internal structure identifies it as this number. Consequently it does not make sense for the structuralist to say that a number is given independently from the other numbers. Once we have fixed the von Neumann ordinals as the coordinate system, we can perhaps speak of algorithms again. We are concerned here, however, with the problem of finding criteria for acceptable coordinate systems.

In order to substantiate these claims, we shall look more closely at the basic motivations for the recursion-theoretic notions. Turing machines, which are usually used for explaining the concept of an algorithm, are somewhat ambiguous objects. Of course, Turing machines can be presented as purely mathematical objects acting on natural numbers. But insofar as Turing machines are really taken to compute, they are presented as objects acting on notations, for instance, on marks on imaginary paper strips. That is, these Turing machines act also on notations rather than on numbers. Since these notations are notations for natural numbers, the difference does not matter for most purposes. We have, however, also an informal concept of computability on other notations. So, in general, algorithms are instructions for manipulating symbols whether they are notations for natural numbers or not.

We have not said what symbols or notations are. At bottom, the concept of symbol is an intentional notion. Whether or not something is a symbol depends on whether we intend to use it to convey meaning. Even concrete objects like apples and chairs could be symbols in this sense. For the present purposes, however, these concepts need not be explained in any detail. Their deeper semantical role is irrelevant for the explication of computability. For our purposes only one fundamental feature matters: Symbols do have an internal structure. They are different because of their inherent properties, graphical properties for instance. They are not just distinguished by their place in a structure.

In the case of practical recursiveness with which we are concerned, what is manipulated *is* symbols.⁸ In the algorithm for division, for instance, we

⁸ Again, this is not to deny that there may be senses of 'practical recursiveness' which do not even presuppose the manipulation of symbols but of still other objects.

must be able to distinguish the symbol '5' from the symbol '6.' If the symbols could not be distinguished, we could not follow the instructions of the algorithm. In other words, the standard numerals are symbols that belong to a notation system. This notation system is indeed a structure. But the place of a numeral in the structure is reflected in the *internal structure* of the symbol. In this sense, the numerals differ from the numbers when platonically conceived. Platonic numbers do not feature an internal structure. Their place in the number structure is imposed from the outside, so to speak. And exactly this is a feature of the platonic natural-number structure which makes it so mysterious.

6. Platonic Computational Structuralism

If the structuralist rules out nonstandard models of arithmetic by invoking REC2, then he cannot understand 'recursive' in the sense of '*theoretically recursive*', because the theoretical notion of recursiveness presupposes mathematical notions that must not be presupposed. Therefore 'recursive' must be understood in the practical sense.

Practical algorithms do not act on natural numbers, but on objects (symbols) with an internal structure. Since natural numbers do not feature an internal structure, these symbols are distinct from the numbers.

Consequently, the coding in REC2 cannot be a coding *in the natural numbers* but a coding by symbols. The only important feature of symbols is that they can be distinguished by their internal structure. If REC2 is understood this way, it does not presuppose the numbers anymore. Therefore the objection that REC2 presupposes notions that are to be defined does not apply anymore. We do have another presupposition. The notion of computability employed is not a mathematical notion, but rather an informal notion. Therefore the claim that any model that can be coded in such a way that addition comes out computable is standard relies on Church's thesis.

Now it might be argued that if we have symbols, we are already presupposing something that is as problematic as the natural numbers themselves, because finitely many symbols will not suffice. But this shows only that the present account is not a nominalistic reduction. However, we did not set out in order to give a nominalistic reconstruction of number theory. Rather, our problem was the opposite: We had many models of first-order arithmetic and asked which one is an intended model. Now the following is suggested as a non-circular answer to this question. *Any model will do provided that we have notations for the elements of the model such that the operations of addition and multiplication are computable on the notations.* Then Tennenbaum's theorem will ensure that the model is standard, that is, that it has order-type ω .

This seems to leave REC2 as an attractive option for banning unintended models, that is, nonstandard models. REC2 was saved as a sensible move

of the structuralist by using a practical notion of recursiveness. Practical recursiveness applies to symbols with internal distinguishing features, not to numbers without such features.

Nevertheless, all is not well with this proposal. A coding in this sense assigns to every object of the model a symbol, and conversely it assigns to every symbol exactly one object. In other words, to put REC2 to work and to judge whether a model is an intended model, we must be able to judge whether some assignment of objects to symbols is a coding. In particular, we have to judge whether every object receives a symbol. But our inability to decide whether all objects are named by some (standard) numeral was our original problem. If we were able to 'see' whether all objects are named by some symbol in some notation system, then we also would be able to see whether all elements in the model are named by some standard numeral. And we are back again to the naive and unacceptable 'A model is intended if all its elements are named by a standard numeral'.

7. Formalist Computational Structuralism

What is wrong with the previous proposal is that the fact that we are computing *on* the natural numbers is not taken seriously enough. For the proposal still allows the elements of intended models to be unstructured objects.

If we take seriously the idea that we compute on the natural numbers, then the numbers must have internal structure. Thus we are driven to a formalist identification of natural numbers with structured linguistic entities. In the spirit of Benacerraf [1965], we do not want to identify the natural-number structure with any single notation system. For we want to say that the Romans, for instance, were calculating with the same numbers as we now do. But as structuralists, we want all intended models to exemplify a unique structure. This leads us to our final proposal:

REC3: Intended models are notation systems with recursive operations on them satisfying the Peano axioms.

This proposal fixes an isomorphism type without admitting *all* systems exemplifying the type to count as intended models. For Tennenbaum's theorem now still guarantees that all intended models are isomorphic.

And we claim that we avoid, in our proposal, all appeal to the natural numbers in fixing the intended models. There no longer is any need to see that all objects are named, for the objects in intended models all *are* names. Our proposal entails that in a fundamental sense, arithmetic is exclusively *about* notations.

According to our proposal, the elements of the structure of natural numbers do not feature any particular internal structure. The structure is

obtained by abstracting from the peculiarities of the respective notation systems. So our position still deserves to be labeled ‘structuralist’.

8. Comparison with Other Structuralist Accounts

In this paper we have put forward a proposal of how someone doing arithmetic can succeed in determining the domain of discourse. We are looking at the situation from a standpoint that includes set theory and everything else that is available to the mathematician. So we can freely talk about nonstandard models, *etc.*

We claim that somebody doing arithmetic *calculates* with the numbers. In particular, the person has effective methods for calculating the sum of any two numbers. His knowledge of the method may be only implicit, and he may lack the notion of finiteness, *etc.* Only we, the observers, make use of these notions. Moreover, we can show that the person’s ability to carry out calculations eliminates the possibility that the person is talking about nonstandard models.

Second-Order Structuralists would rely on the categoricity of second-order arithmetic in order to rule out nonstandard models. However, the person doing arithmetic may simply not make use of second-order quantifiers. We think the calculation of sums is far more basic to arithmetic than claims about sets of numbers. In this sense our approach relies on a feature that is much more intrinsic to arithmetic than the use of second-order quantifiers.

In another sense, Second-Order Structuralism may be thought to be more basic than Computational Structuralism. The Second-Order Structuralist relies on the categoricity of second-order arithmetic, which is in a sense more straightforward than Tennenbaum’s theorem. The additional expenditures of Computational Structuralism, however, are only technical and not philosophical. We are focusing on a person’s abilities to do arithmetic and try to argue that they are sufficient to rule out nonstandard models. On the account of Computational Structuralism, it may require more effort on *our* side to see why the person does not talk about nonstandard models. The assumptions on the abilities can be kept to a bare minimum in return; he must be able to carry out additions. The Second-Order Structuralist, in contrast, has to assume that the person must somehow grasp second-order quantification or a sufficiently large fragment thereof.

The observation that our proposal makes only very weak assumptions on the abilities of the person who is doing arithmetic requires a formal qualification. The second-order strategy for ruling out nonstandard models relies very much on a strengthening of the induction principle. Basically one obtains a theory that can determine its models up to isomorphism by allowing many more conditions in induction than just those that can be described by arithmetical conditions. So far our approach does not seem to

make much use of induction at all. It seems to rely on the computability of the sum of any two given numbers. However, it does not rely exclusively on this. We also need *some* amount of induction in order to banish nonstandard models because Tennenbaum's theorem cannot be proved for systems without induction: McAloon [1982] showed that the operations of addition and multiplication are not recursive in any nonstandard model of $I\Delta_0$. But we definitely do not need any kind of *second-order* induction or the extendability of the induction scheme to formulas with new vocabulary, because Tennenbaum's theorem holds for Peano arithmetic. Indeed, very little induction is required indeed for Computational Structuralism.⁹ For our purposes it is sufficient that the first-order induction axioms of Peano arithmetic are more than enough.

In another respect Second-Order Structuralism scores higher than Computational Structuralism. The former applies not only to arithmetic; second-order quantification can be used in order to rule out nonstandard models of set theory as well. Computational Structuralism, in contrast, seems to be restricted to arithmetic. We cannot think of any sensible restrictions on the computability of set-theoretic operations, and it is unclear what an analogue of Tennenbaum's theorem for set theory could be. So we do not have much to say to the question how other mathematical structures, such as the real numbers, for example, are fixed. Perhaps arithmetic's extremely close connection to our practices of computing makes it a very special domain.

The limited scope of the Computational Structuralist approach supports our intuition that we know much better what we are talking about when we are doing arithmetic than when we are doing set theory. At least we are more disposed to admit that we cannot so easily single out the intended models of set theory than to admit that arithmetic is also about nonstandard models. Therefore the limited scope of application of Computational Structuralism may not be a fault at all.

⁹ For more information on how much induction is needed see D'Aquino [1997] and Kaye [1993].

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