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Two Proof-Theoretic Remarks on EA + ECT

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Abstract. In this note two propositions about the epistemic formalization of Church's Thesis (ECT) are proved. First it is shown that all arithmetical sentences deducible in Shapiro's system EA of Epistemic Arithmetic from ECT are derivable from Peano Arithmetic PA + *uniform reflection* for PA. Second it is shown that the system EA + ECT has the epistemic disjunction property and the epistemic numerical existence property for arithmetical formulas.

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1 Introduction

EA + ECT is a formal theory which has been considered in the investigation of epistemic systems of arithmetic. The system EA + ECT consists of SHAPIRO's system EA of Epistemic Arithmetic (see [9]) plus the following schematic epistemic formalization ECT of CHURCH's Thesis (see [2]):

$$\Box\forall x\exists y\,\Box A(x,y) \;\rightarrow\; \exists e\forall x\exists y\,[T(e,x,y)\;\wedge\;A(x,U(y))].$$

Here A ranges over sentences of the language $\mathcal{L}_{\mathrm{EA}}$ of EA, T is KLEENE's T-predicate and U is KLEENE's U-function symbol. It has been known for some time that theory EA + ECT is consistent. Proofs for this fact (in decreasing order of complexity) are given in [2, 7, 8]. But one would like to have more detailed information concerning the arithmetical strength of EA + ECT. Such information cannot be directly extracted from the existing consistency proofs of EA + ECT.

Using a variation on the method of the KLEENE slash, SHAPIRO showed in [9] that EA has the following epistemic analogue of the disjunction property EDP and the numerical existence property ENEP:

1. For all sentences A, $B \in \mathcal{L}_{\mathrm{EA}}$: if $\mathrm{EA} \vdash (\Box A \vee \Box B)$, then $\mathrm{EA} \vdash \Box A$ or $\mathrm{EA} \vdash \Box B$.
2. For all formulas $A(x) \in \mathcal{L}_{\mathrm{EA}}$: if $\mathrm{EA} \vdash \exists x\,\Box A(x)$, then there is a natural number n such that $\mathrm{EA} \vdash \Box A(\overline{n})$.

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Flagg lists EDP and ENEP as natural conditions that any epistemic framework must meet in order to serve as a reasonable synthesis of classical and constructivistic mathematics (see [3, p. 27/28]). Unfortunately, it appears that Shapiro's method cannot be used to show that EA + ECT has EDP and ENEP.

Therefore in this note we are concerned with the following two questions:

1. What is the arithmetical strength of EA + ECT ?
2. Does EA + ECT have EDP and ENEP ?

Partial answers are provided to these questions. With respect to the first question, an upper bound to the arithmetical strength of consequences of ECT in the context of EA is given: all arithmetical sentences deducible in EA from ECT belong to PA + *uniform reflection for* PA. With respect to the second question, it is shown that EA + ECT has EDP and ENEP for *arithmetical* formulas, i. e.,

1. For all sentences A, $B \in \mathcal{L}_{\mathrm{PA}}$: if $\mathrm{EA}+\mathrm{ECT} \vdash (\Box A \vee \Box B)$, then $\mathrm{EA}+\mathrm{ECT} \vdash \Box A$ or $\mathrm{EA} + \mathrm{ECT} \vdash \Box B$.

2. For all formulas $A(x) \in \mathcal{L}_{\mathrm{PA}}$: if $\mathrm{EA} + \mathrm{ECT} \vdash \exists x\, \Box A(x)$, then there is a natural number n such that $\mathrm{EA} + \mathrm{ECT} \vdash \Box A(\overline{n})$.

In the sequel, for any theory S we will be working with in the sequel Bew_S designates the standard provability of S. A, B etc. will be used as variables ranging over formulas. If A has the free variables $x_1, \ldots, x_n$, then $\mathrm{Bew}_S(A)$ expresses that A is S-provable if the variable x_i is replaced by the numeral for x_i for $i \leq n$. Thus in $\mathrm{Bew}_S(A)$ the free variables of A may be bound from outside. Similar convnetions apply if free variables are explicitly mentioned as in $A(x, y)$.

2 The arithmetical strength of ECT

Consider the following schematic principle RC: $\Box A \rightarrow \mathrm{Bew}_{\mathrm{EA}}(A)$. We will first translate EA + RC to PA using the following translation $\sigma : \mathcal{L}_{\mathrm{EA}} \longrightarrow \mathcal{L}_{\mathrm{PA}}$:

- if A is atomic, then $\sigma(A) \equiv A$;
- σ distributes over the predicate logical connectives;
- $\sigma(\Box A) \equiv \mathrm{Bew}_{\mathrm{EA}}(A) \wedge \sigma(A)$.

Since σ is primitive recursive, we can work freely with σ in PA.

Lemma 1. *For all* $A \in \mathcal{L}_{\mathrm{EA}}$*: if* $\mathrm{EA} + \mathrm{RC} \vdash A$*, then* $\mathrm{PA} \vdash \sigma(A)$*. Moreover, the proof of this assertion is formalizable in* PA*.*

Proof. Clearly, $\mathrm{PA} \vdash \sigma(\Box A \rightarrow \mathrm{Bew}_{\mathrm{EA}}(A))$ ($\equiv \mathrm{Bew}_{\mathrm{EA}}(A) \wedge \sigma(A) \rightarrow \mathrm{Bew}_{\mathrm{EA}}(A)$). Similarly, $\mathrm{PA} \vdash \sigma(\Box(\Box A \rightarrow \mathrm{Bew}_{\mathrm{EA}}(A)))$. Also the σ-translations of all axioms of EA are provable in PA. So σ translates EA + RC-proofs into PA-proofs. This argument can evidently be carried out in PA. □

Now we define *uniform reflection* for PA and EA in the standard way:

$$\mathrm{REFL}_{\mathrm{PA}} \equiv \mathrm{Bew}_{\mathrm{PA}}(A) \rightarrow A, \quad \text{if } A \in \mathcal{L}_{\mathrm{PA}};$$
$$\mathrm{REFL}_{\mathrm{EA}} \equiv \mathrm{Bew}_{\mathrm{EA}}(A) \rightarrow A, \quad \text{if } A \in \mathcal{L}_{\mathrm{EA}}.$$

As indicated above, A may contain free variables, that are also free in $\mathrm{Bew}_{\mathrm{PA}}(A)$ and $\mathrm{Bew}_{\mathrm{EA}}(A)$.

Lemma 2. $\mathrm{REFL}_{\mathrm{EA}}$ *is consistent with* EA + RC.

Proof. Suppose we had a derivation $\mathcal{P}$ in EA + RC of a contradiction from instances of $\mathrm{REFL}_{\mathrm{EA}}$. Then $\sigma(\mathcal{P})$ is a proof of a sentence

$$(\mathrm{Bew}_{\mathrm{EA}}(A_1) \to \sigma(A_1)) \wedge \cdots \wedge (\mathrm{Bew}_{\mathrm{EA}}(A_n) \to \sigma(A_n)) \to 0 = 1$$

in PA. Since, for all $i \leq n$, $\mathrm{PA} \vdash \mathrm{Bew}_{\mathrm{EA}}(A_i) \to \mathrm{Bew}_{\mathrm{EA}}(\sigma(A_i))$, this can then be transformed into a proof in PA of

$$(\mathrm{Bew}_{\mathrm{EA}}(\sigma(A_1)) \to \sigma(A_1)) \wedge \cdots \wedge (\mathrm{Bew}_{\mathrm{EA}}(\sigma(A_n)) \to \sigma(A_n)) \to 0 = 1,$$

i. e. a PA-proof of the inconsistency of $\mathrm{REFL}_{\mathrm{EA}}$. But there is no such proof. Indeed, it is easy to see that $\mathrm{Bew}_{\mathrm{EA}}(\sigma(A_i)) \to \sigma(A_i)$ is true for every (arithmetical) $\sigma(A_i)$. Erasing all occurrences of $\Box$ in an EA-proof of $\sigma(A_i)$ yields a PA-proof of $\sigma(A_i)$, whereby $\sigma(A_i)$ must be true. $\Box$

The consistency of $\mathrm{EA} + \mathrm{RC} + \mathrm{REFL}_{\mathrm{EA}}$ does not follow from this proof, because in $\mathrm{EA} + \mathrm{RC} + \mathrm{REFL}_{\mathrm{EA}}$ the rule of necessitation may be applied to any theorem of the theory including those proved by appeal to instances of RC and $\mathrm{REFL}_{\mathrm{EA}}$.

Lemma 3. *Let C be any instance of* ECT. *Then there is an instance R of* $\mathrm{REFL}_{\mathrm{EA}}$ *such that* $\mathrm{EA} + \mathrm{RC} \vdash R \to C$.

Proof. Assume the antecedent $\Box\forall x \exists y\, \Box A(x, y)$ of C. From this we infer, using RC, that

(1) $\forall x \exists y\, \mathrm{Bew}_{\mathrm{EA}}(A(x, y))$.

If $B_{\mathrm{EA}}(u, z)$ expresses that u is an EA-proof for z, then (1) is defined as

$$\forall x \exists y \exists u\, B_{\mathrm{EA}}(u, A(x, y)).$$

From this we get $\forall x \exists w (\exists y \leq w)(\exists u \leq w)\big(w = \langle u, y\rangle \wedge B_{\mathrm{EA}}(u, A(x, y))\big)$. There is a recursive function $\{e_1\}$ with index e_1 giving applied to a number x the smallest pair $\langle u, y\rangle$ such that u is an EA-proof of $A(x, y)$. Thus we have,

$$\forall x \exists z (\exists y \leq z)(\exists u \leq z)\big(T(e_1, x, z) \wedge U(z) = \langle u, y\rangle \wedge B_{\mathrm{EA}}(u, A(x, y))\big).$$

This implies also the following:

$$\forall x \exists z (\exists y \leq z)(\exists u \leq z)\big(T(e_1, x, z) \wedge U(z) = \langle u, y\rangle \wedge \mathrm{Bew}_{\mathrm{EA}}(A(x, y))\big).$$

From the index e_1 we get another index e such that $\{e\}(x)$ is the first coordinate of the pair $\{e_1\}(x)$, if it exists. Thus we arrive at

$$\forall x \exists z (\exists y \leq z)(\exists u \leq z)\big(T(e, x, z) \wedge U(z) = y \wedge \mathrm{Bew}_{\mathrm{EA}}(A(x, y))\big).$$

Now, using $\mathrm{Bew}_{\mathrm{EA}}(A(x, y)) \to A(x, y)$, we infer to

$$\exists z\big(T(e, x, z) \wedge A(x, y) \wedge U(z) = y\big).$$ $\Box$

Theorem 4. *If* $\mathrm{EA} \vdash C_1 \wedge \cdots \wedge C_n \to B$, *with* $C_1, \ldots, C_n$ *instances of* ECT *and B arithmetical, then* $\mathrm{PA} \vdash R_1 \wedge \cdots \wedge R_n \to B$ *for some instances* $R_1, \ldots, R_n$ *of* $\mathrm{REFL}_{\mathrm{PA}}$.

Proof. If $\mathrm{EA} \vdash C_1 \wedge \cdots \wedge C_n \to B$, then, by Lemma 3,

$$\mathrm{EA} + \mathrm{RC} \vdash R_1^* \wedge \cdots \wedge R_n^* \to B$$

for some instances $R_1^*, \ldots, R_n^*$ of $\mathrm{REFL}_{\mathrm{EA}}$. By Lemma 1, this proof can be transformed into a PA-proof of

$$(\mathrm{Bew}_{\mathrm{EA}}(A_1) \to \sigma(A_1)) \wedge \cdots \wedge (\mathrm{Bew}_{\mathrm{EA}}(A_n) \to \sigma(A_n)) \to B,$$

for some $A_1, \ldots, A_n$. But by Lemma 1, for all i, $\mathrm{PA} \vdash \mathrm{Bew}_{\mathrm{EA}}(A_i) \to \mathrm{Bew}_{\mathrm{EA}}(\sigma(A_i))$. PA also proves $\mathrm{Bew}_{\mathrm{EA}}(\sigma(A_i)) \to \mathrm{Bew}_{\mathrm{PA}}(\sigma(A_i))$ by formalizing the argument that the "eraser"-translation (which removes all occurrences of $\Box$ from a formula of $\mathcal{L}_{\mathrm{EA}}$) translates EA-proofs into PA-proofs. This gives us the desired result. $\Box$

Corollary 5. *All arithmetical sentences that are deducible in* EA *from* ECT *belong to* $\mathrm{PA} + \mathrm{REFL}_{\mathrm{PA}}$.

It is an open question whether this result still holds when ECT is not used as a hypothesis, but instead is added as a new axiom, yielding the theory EA + ECT. Corollary 5 is nevertheless not devoid of philosophical significance, because CHURCH's Thesis is usually regarded as being for quasi-empirical reasons extremely plausible, but not having the same status as, e. g., axioms of mathematical induction. In short, when it is used in mathematical arguments, it is used as an hypothesis (as the axiom of choice once was). We also do not know whether the upper bound on the arithmetical strength of ECT can be improved, e. g. whether the statement of the corollary is still true if we replace $\mathrm{PA} + \mathrm{REFL}_{\mathrm{PA}}$ by PA.

3 EDP and ENEP for arithmetical formulas

We begin by introducing a translation function $\tau : \mathcal{L}_{\mathrm{EA}} \longrightarrow \mathcal{L}_{\mathrm{EA}}$ defined as follows:

- if A is atomic, then $\tau(A) \equiv A$;
- τ distributes over the predicate logical connectives;
- $\tau(\Box A) \equiv \Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \Box\tau(A)$.

Here $\mathrm{Bew}_{\mathrm{EA+ECT}}$ is of course the standard provability predicate for EA + ECT. KLEENE's recursion theorem is used to show that the translation τ is well-defined.

As a first observation, it is noted that τ is a *sound* translation from EA + ECT to EA + ECT:

Lemma 6. *If* $\mathrm{EA} + \mathrm{ECT} \vdash A$, *then* $\mathrm{EA} + \mathrm{ECT} \vdash \tau(A)$.

Proof by induction on the length of proofs in EA + ECT.

(i) The logical and arithmetical axioms present no problems, since τ distributes over the logical connectives.

(ii) $\tau(\Box A \rightarrow A) \equiv \tau(\Box A) \rightarrow \tau(A) \equiv \Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \Box\tau(A) \rightarrow \tau(A)$,

and this is provable in EA + ECT.

(iii)
$$\begin{aligned}\tau(\Box A \rightarrow \Box\Box A) &\equiv \tau(\Box A) \rightarrow \tau(\Box\Box A)\\ &\equiv \Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \Box\tau(A) \rightarrow \Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(\Box A)) \wedge \Box\tau(\Box A)\\ &\equiv \Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \Box\tau(A)\\ &\quad \rightarrow \Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \Box\tau(A))\\ &\qquad \wedge \Box(\Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \Box\tau(A)).\end{aligned}$$

By derivability conditions for $\mathrm{Bew}_{\mathrm{EA+ECT}}$ and epistemic laws governing $\Box$, this is seen to be provable in EA + ECT.

(iv)
$$\begin{aligned}\tau(\Box A \rightarrow (\Box(A \rightarrow B) \rightarrow \Box B)) &\equiv \tau(\Box A) \rightarrow (\tau(\Box(A \rightarrow B)) \rightarrow \tau(\Box B))\\ &\equiv \Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \Box\tau(A)\\ &\quad \rightarrow (\Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A) \rightarrow \tau(B)) \wedge \Box(\tau(A) \rightarrow \tau(B))\\ &\qquad \rightarrow \Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(B)) \wedge \Box\tau(B)).\end{aligned}$$

By derivability conditions for $\mathrm{Bew}_{\mathrm{EA+ECT}}$ and epistemic laws governing $\Box$, this is easily seen to be provable in EA + ECT.

(v) Suppose $\mathrm{EA}+\mathrm{ECT} \vdash \tau(A)$. Then $\mathrm{EA}+\mathrm{ECT} \vdash \Box\mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \Box\tau(A)$, and since $\Box\mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \tau(A) \equiv \tau(\Box A)$, it follows that $\mathrm{EA}+\mathrm{ECT} \vdash \tau(\Box A)$.

(vi) ECT is treated as follows. The translation of the antecedent is

$$\begin{aligned}&\Box\mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(\forall x\exists y\, \Box A(x,y))) \wedge \Box\tau(\forall x\exists y\, \Box A(x,y))\\ &\qquad\equiv \Box\mathrm{Bew}_{\mathrm{EA+ECT}}(\forall x\exists y\, (\Box\mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A(x,y))) \wedge \Box\tau(A(x,y))))\\ &\qquad\quad \wedge \Box\forall x\exists y\, (\Box\mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A(x,y))) \wedge \Box\tau(A(x,y))).\end{aligned}$$

The consequent of ECT is translated as

$$\exists e\forall x\exists y\, \big(T(e,x,y) \wedge \Box\mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A(x,U(y)))) \wedge \Box(\tau(A(x,U(y))))\big).$$

Applying ECT to the second conjunct of the translation of the antecedent yields

$$\exists e\forall x\exists y\, \big(T(e,x,y) \wedge \Box\mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A(x,U(y)))) \wedge \Box(\tau(A(x,U(y))))\big).$$

But this is exactly the translation of the consequent of ECT.

(vii) Modus ponens is trivial. □

Next we want to prove that $\mathrm{EA}+\mathrm{ECT}$ is Σ_1-correct for arithmetical statements. This statement is slightly stronger than the well-known fact that $\mathrm{EA}+\mathrm{ECT}$ is consistent. Using an old theorem of FRIEDMAN, this proposition can be extracted from a known proof of the consistency of $\mathrm{EA}+\mathrm{ECT}$.

The system $\mathrm{HA}+\mathrm{ICT}$ consists of Heyting arithmetic HA plus the following *intuitionistic version* ICT of CHURCH's Thesis (see [10, p. 195]):

$$\forall x\exists y\, A(x,y) \to \exists e\forall x\exists y\, \big(T(e,x,y) \wedge A(x,U(y))\big).$$

Here A ranges over formulas of $\mathcal{L}_{\mathrm{HA}}$.

Lemma 7 (FRIEDMAN [4], see also [1, p. 397/398].) $\mathrm{HA}+\mathrm{ICT}$ *is closed under Markov's Rule for primitive recursive parameters.* □

Using FRIEDMAN's theorem, a trick of [8] can be used to show that $\mathrm{EA}+\mathrm{ECT}$ is Σ_1-correct for arithmetical statements. This trick makes use of the notion of *constructivization of a proof* (see [8, p. 652]), which is defined as follows. First, the reader is reminded of the Gödel-translation $g : \mathcal{L}_{\mathrm{HA}} \longrightarrow \mathcal{L}_{\mathrm{EA}}$:[4)]

if A is atomic, then $g(A) \equiv \Box A$;

$g(A \circ B) \equiv \Box g(A) \circ \Box g(B)$ for $\circ \in \{\wedge, \vee, \to, \leftrightarrow\}$; $g(\neg A) \equiv \Box\neg\Box g(A)$;

$g(\forall x\, A(x)) \equiv \Box\forall x\, g(A(x))$; $g(\exists x\, A(x)) \equiv \exists x\, \Box g(A(x))$.

Moreover let $A^{-\Box}$ be the result of removing all occurrences of $\Box$ from a formula A. Then the *constructivization of a formula* $A \in \mathcal{L}_{\mathrm{EA}}$ is defined to be $g(A^{-\Box})$. The *constructivization of a proof* results from replacing each sentence in the proof by its constructivization.

Lemma 8. $\mathrm{EA}+\mathrm{ECT}$ *is* Σ_1*-correct for arithmetical statements.*

Proof. Suppose $\mathrm{EA}+\mathrm{ECT} \vdash \exists x\, A$ with A arithmetical Δ_0. Then $\mathrm{HEA}+\mathrm{ECT}$ $\neg\neg\exists x\, A$, where HEA is the constructive fragment of EA. Consider the constructivization of this proof. This can be considered as a proof of $\neg\neg\exists x\, A$ in $\mathrm{HA}+\mathrm{ICT}$. By FRIEDMAN's theorem, $\mathrm{HA}+\mathrm{ICT}$ then proves $\exists x\, A$. But $\mathrm{HA}+\mathrm{ICT}$ is ω-consistent ([10, p. 196/197]. Therefore, $\exists x\, A$ must be classically true in the standard model of arithmetic. □

[4)] This translation function was first introduced by GÖDEL [6] in the context of propositional logic. For a discussion of GÖDEL's translation in the context of Epistemic Arithmetic, see [9, p. 24/25].

Now we have all the necessary ingredients for the proof of EDP for arithmetical formulas for EA + ECT:

T h e o r e m 9. *For all sentences* $A, B \in \mathcal{L}_{\mathrm{PA}}$, *if* $\mathrm{EA} + \mathrm{ECT} \vdash (\Box A \vee \Box B)$, *then either* $\mathrm{EA} + \mathrm{ECT} \vdash A$ *or* $\mathrm{EA} + \mathrm{ECT} \vdash B$.

P r o o f. Suppose $\mathrm{EA} + ECT \vdash (\Box A \vee \Box B)$. Then by Lemma 6 we have that $\mathrm{EA} + \mathrm{ECT} \vdash \tau(\Box A \vee \Box B)$. But

$$\begin{aligned} \tau(\Box A \vee \Box B) &\equiv \tau(\Box A) \vee \tau(\Box B) \\ &\equiv (\Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(A)) \wedge \Box\tau(A)) \vee (\Box \mathrm{Bew}_{\mathrm{EA+ECT}}(\tau(B)) \wedge \Box\tau(B)), \end{aligned}$$

where $\tau(A) \equiv A$ and $\tau(B) \equiv B$, because $A, B \in \mathcal{L}_{\mathrm{PA}}$. Therefore this entails

$$\mathrm{EA} + \mathrm{ECT} \vdash \mathrm{Bew}_{\mathrm{EA+ECT}}(A) \vee \mathrm{Bew}_{\mathrm{EA+ECT}}(B).$$

By the Σ_1-correctness of EA + ECT for arithmetical statements, it then follows that either $\mathrm{EA} + \mathrm{ECT} \vdash A$ or $\mathrm{EA} + \mathrm{ECT} \vdash B$. □

It can be established in a similar way that EA + ECT has ENEP for arithmetical formulas. Alternatively, one can appeal to FRIEDMAN and SHEARD's theorem that the epistemic disjunction property and the epistemic numerical existence property are equivalent in Epistemic Arithmetic (see [5]).

It remains an open question whether EA + ECT has EDP and ENEP for *all* formulas of the language of Epistemic Arithmetic.

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