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NO FUTURE

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ABSTRACT. The difficulties with formalizing the intensional notions *necessity, knowability* and *omniscience,* and *rational belief* are well-known. If these notions are formalized as predicates applying to (codes of) sentences, then from apparently weak and uncontroversial logical principles governing these notions, outright contradictions can be derived. Tense logic is one of the best understood and most extensively developed branches of intensional logic. In tense logic, the temporal notions *future* and *past* are formalized as sentential operators rather than as predicates. The question therefore arises whether the notions that are investigated in tense logic can be consistently formalized as predicates. In this paper it is shown that the answer to this question is negative. The logical treatment of the notions *of future* and *past* as predicates gives rise to paradoxes due the specific *interplay* between both notions. For this reason, the tense paradoxes that will be presented are not identical to the paradoxes referred to above.

KEY WORDS: tense logic, tense predicates, diagonalization, paradox

1. THE PARADOXES

The derivations of the known paradoxes concerning necessity, knowability and rational belief are *diagonal arguments* that resemble the derivation of the Liar paradox. These derivations all make essential use of the T axiom scheme $\Box\varphi \rightarrow \varphi$ of modal logic. Montague's paradox uses the axiom $N\overline{\varphi} \rightarrow \varphi$, where N is a necessity *predicate* and $\overline{\varphi}$ is the numeral denoting the Gödel number of the formula φ. Similarly, the so-called *paradox of the knower* due to Kaplan and Montague uses the corresponding principle $K\overline{\varphi} \rightarrow \varphi$, where K is a knowability predicate. Thomason's paradox about rational belief makes use of the principle $R\overline{R(\overline{\varphi} \rightarrow \varphi)}$, where R is a rational belief predicate. This latter principle is a weakening of the T axiom. Thus one obtains only an "internal" inconsistency, i.e. one cannot derive $\bot$ but only $R\overline{\bot}$.

In the classical systems of (propositional) tense logic the notions of *future* and *past* are investigated. The language of propositional tense logic contains, aside from the usual Boolean propositional connectives and a stock of propositional variables, two sentential operators $\underline{G}$, $\underline{H}$, of which

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the intended interpretation is 'at every time in the future' and 'at every time in the past', and their duals $\underline{F}$, $\underline{P}$ which may be read as 'at some time in the future' and 'at some time in the past', respectively. The familiar systems of tense logic do not contain a version of T as an axiom. It is therefore not initially obvious whether a diagonal argument can be set up which generates a paradox for tense logical systems in which temporal operators are replaced by tense predicates.[1] Nevertheless a paradox can be produced for reasonable tense logics which treat tense notions as predicates. This is unfortunate, because the replacement of operators by predicates would enable us to express quantifications over an infinite set of sentences in a straight forward way, which would prima facie be an attractive feature.

Assume a first-order language $\mathcal{L}$ which contains unary predicates G, H, F, P such that $G(H)$ is true of φ if φ will be (was) true at every moment in the future (past), $F(P)$ is true of φ if φ will be (was) true at some moment in the future (past).[2] Furthermore, $\mathcal{L}$ is assumed to contain the standard language of arithmetic, so that it can talk and reason about its own syntax (via coding).

We will consider the following systems $\mathbf{K}_t^*$ and $\mathbf{S}$, which are formulated in $\mathcal{L}$. $\mathbf{K}_t^*$ and $\mathbf{S}$ both contain the axioms of predicate logic with identity, and the axioms of Robinson's system of arithmetic $\mathbf{Q}$.[3] In $\mathbf{Q}$, all recursive functions are representable. Therefore $\mathbf{Q}$ can prove Gödel's diagonal lemma.[4]

Furthermore, $\mathbf{K}_t^*$ contains all instances of the axiom schemes G1–3 and H1–3:

G1 $G\overline{\varphi \rightarrow \psi} \rightarrow (G\overline{\varphi} \rightarrow G\overline{\psi})$,
H1 $H\overline{\varphi \rightarrow \psi} \rightarrow (H\overline{\varphi} \rightarrow G\overline{\psi})$,
G2 $\varphi \rightarrow H\overline{F\overline{\varphi}}$,
H2 $\varphi \rightarrow G\overline{P\overline{\varphi}}$,
G3 $G\overline{\varphi} \leftrightarrow \neg F\overline{\neg\varphi}$,
H3 $H\overline{\varphi} \leftrightarrow \neg P\overline{\neg\varphi}$
(for all $\varphi, \psi \in \mathcal{L}$).

$\mathbf{S}$ contains all instances of G2, G3, H3 and the scheme D*:

D* $H\overline{\varphi} \rightarrow P\overline{\varphi}$
(for all $\varphi \in \mathcal{L}$).

Apart from the axioms given, $\mathbf{K}_t^*$ and $\mathbf{S}$ have no other axioms. Both systems are closed under Modus Ponens and the following two rules of inference:

R1 $\vdash\varphi \Rightarrow \vdash G\overline{\varphi}$,
R2 $\vdash\varphi \Rightarrow \vdash H\overline{\varphi}$
(for all $\varphi \in \mathcal{L}$).

The propositional counterpart to $\mathbf{K}_t^*$ is the minimal tense logic $\mathbf{K}_t$,[5] which seems unobjectionable. The propositional counterpart to $\mathbf{S}$ is a proper fragment of $\mathbf{K}_t$ plus a well-known axiom which is not implied by $\mathbf{K}_t$.[6] It says, loosely, that if something has always been the case, then it has been the case at some time in the past. This statement does embody a substantial assumption concerning the structure of time, namely that time does not have a first moment.[7]

Instead of employing axiom *schemes* for $\mathbf{K}_t^*$ and for $\mathbf{S}$, we might as well have used single *axioms* quantifying over (codes of) sentences; but, as will be seen, specific instances of the schemes above suffice to derive the paradoxes.

1.1. *The Internal Inconsistency of* $\mathbf{K}_t^*$

Since $\mathbf{Q}$ proves Gödel's diagonal lemma, and $\mathbf{Q}$ is contained in $\mathbf{K}_t^*$, one can find sentences α, β of $\mathcal{L}$ s.t.

- $\mathbf{K}_t^* \vdash \alpha \leftrightarrow H\overline{F\overline{\neg\alpha}}$,
- $\mathbf{K}_t^* \vdash \beta \leftrightarrow G\overline{P\overline{\neg\beta}}$.

Now α and β are used to prove that no instant of time has a future or a past,[8] i.e.:

THEOREM 1.1. $\mathbf{K}_t^* \vdash H\overline{\bot} \wedge G\overline{\bot}$.

Therefore we say that $\mathbf{K}_t^*$ is *internally inconsistent*. We devide the proof of Theorem 1.1. into two subproofs:

LEMMA 1.2. $\mathbf{K}_t^* \vdash H\overline{\bot}$.

Proof. In $\mathbf{K}_t^*$, suppose for reductio that $\neg\alpha$. Then $\neg H\overline{F\overline{\neg\alpha}}$. Therefore, by G2, α follows. Contradiction. So $\mathbf{K}_t^* \vdash \alpha$. Thereby also (i) $\mathbf{K}_t^* \vdash H\overline{F\overline{\neg\alpha}}$. But since $\mathbf{K}_t^* \vdash \alpha$, R1 yields $G\overline{\alpha}$ and thus $\neg F\overline{\neg\alpha}$ by G3. But now R2 yields $H\overline{\neg F\overline{\neg\alpha}}$. I.e., we have both $H\overline{F\overline{\neg\alpha}}$ and $H\overline{\neg F\overline{\neg\alpha}}$, and therefore $\mathbf{K}_t^* \vdash H\overline{\bot}$. □

LEMMA 1.3. $\mathbf{K}_t^* \vdash G\overline{\bot}$.

Proof. For symmetry reasons (β is the temporal mirror image of α). □

The proof of the internal inconsistency of $\mathbf{K}_t^*$ cannot be strengthened to a proof showing the inconsistency of $\mathbf{K}_t^*$, since if the extensions of G and H

are set identical to $\mathcal{L}$ (and the extensions of F and P are set identical to $\emptyset$), a model of $\mathbf{K}_t^*$ is trivially obtained.

1.2. *The Inconsistency of* S

Using α one obtains:

THEOREM 1.4. $\mathbf{S} \vdash \bot$.

Proof. Just as before, derive (i) $\mathbf{S} \vdash H\overline{F\overline{\neg\alpha}}$, and also $H\overline{\neg F\overline{\neg\alpha}}$. Finally, D* is used to obtain $P\overline{\neg F\overline{\neg\alpha}}$, and we have $\neg H\overline{F\overline{\neg\alpha}}$ by H3; therefore, with (i), $\mathbf{S} \vdash \bot$. □

2. Discussion of the Paradoxes

2.1. *Strengthenings of the Liar Paradox*

It is well-known[9] that adding the Tarski biconditional scheme $T\overline{\varphi} \leftrightarrow \varphi$ to weak systems of arithmetic yields an inconsistency. This is of course the formalized version of the Liar paradox. Essentially, the paradoxes of Kaplan and Montague[10] result from assuming only the left-to right direction of the Tarski biconditional scheme (the so-called Reflexivity Axiom $T\overline{\varphi} \rightarrow \varphi$) outright. The converse direction of the Tarski biconditional scheme is weakened by Kaplan and Montague to a rule (the so-called Necessitation Rule $\varphi / T\overline{\varphi}$). In this sense, their paradoxes can be considered as strengthenings of the Liar paradox. Thomason[11] has in a sense strengthened the Liar paradox even further. He has essentially shown that even if one also weakens the Reflexivity Axiom to $T\overline{T\overline{\varphi} \rightarrow \varphi}$ (and in addition assumes the Transitivity Axiom $T\overline{\varphi} \rightarrow T\overline{T\overline{\varphi}}$), an internal inconsistency follows.

It is clear that the "pure past" fragment $\mathbf{S}_P$ of $\mathbf{K}_t^* \cup \mathbf{S}$, consisting of $\mathbf{Q}$ plus H1, H3, D* and R2 is consistent, and indeed internally consistent. It is easily seen that $\mathbf{S}_P$ is a subtheory of Friedman and Sheard's system $\mathbf{F}$, as it is described in Sheard [8, pp. 1048–1049]; $\mathbf{F}$ is known to be consistent and internally consistent.[12] Analogously, the "pure future" fragment of $\mathbf{K}_t^*$ is consistent in any respect, even if an axiom analogous to D* is added. But if $\mathbf{S}_P$ were strengthened by the Transitivity Axiom $H\overline{A} \rightarrow H\overline{H\overline{A}}$, we would have an inconsistency again.[13] McGee [5] has shown that extending $\mathbf{S}_P$ by the Barcan formula, which can be seen as a formalized omega-rule, leads to omega-inconsistency. The temporal paradoxes discussed in this paper concern systems that contain *neither* the transitivity scheme *nor* the Barcan formula. All these inconsistency results can be considered as further strengthenings of the Liar paradox.

2.2. *Philosophical Significance of the Temporal Paradoxes*

The perceived philosophical significance of Montague's paradox about necessity derives from the fact that it concerns the associated predicate system of a plausible and time-honored system of propositional modal logic. Montague's paradox pertains to the predicate logical analogue of Feys' system **T** of modal logic. **T** was one of the first systems of modal logic to be axiomatized and has remained one of the most basic systems of modal logic ever since. It is scarcely an exaggeration to say that **T** gives an exhaustive list of the uncontroversial principles of modal logic, although it is of course an extension of the minimal modal logic **K**. Similar remarks can be made for Kaplan and Montague's epistemic paradox. This epistemic paradox concerns the associated predicate logic of a plausible and basic propositional system for formalizing the "absolute" notion of knowability. With respect to Thomason's paradox concerning rational belief we want to express some reservations as to its philosophical significance. For it is not immediately clear how the axiom $R\overline{R\overline{\varphi} \rightarrow \varphi}$ ("For any sentence φ of the language, it is rational for X to believe that: if it is rational for X to believe φ, then φ") can be convincingly argued for as being intuitively valid.

Now we see that in tense logic the situation is even worse than in modal logic. Our results show that the associated predicate system $\mathbf{K}_t^*$ of Lemmon's minimal tense logic $\mathbf{K}_t$ is internally inconsistent, whereas in the case of modal logic, at least $\mathbf{K}^*$, the associated predicate system of **K**, is consistent. The reason for this difference is the higher degree of expressiveness which the language of tense logic has, compared to the language of usual modal logic. In tense logic, one cannot only express in the object language some properties of the accessibility relation, as in the modal logic of necessity, but also some properties of the *converse* of the accessibility relation and of the relation between these two relations. This explains why the logical interaction between the temporal predicates is essential to our derivations of the paradoxes.

Summing up, the distinctive features of the temporal paradoxes are:

1. $\mathbf{K}_t^*$ is internally inconsistent, whereas the predicate version of **T** is inconsistent,
2. the temporal paradoxes do not completely parallel Montague's paradox,
3. the temporal paradoxes do not affect the pure future part, or the pure past part of tense logic alone.

NOTES

[1] In recent years tense logical languages have been proposed which contain operators for which it *is* obvious that if they are treated as predicates, paradox ensues. For instance, one can investigate the notion 'until (and including) now'. For this notion, the T axiom scheme and the axioms of the minimal modal logic **K** seem clearly valid. Then the argument of the paradoxes of Montague and Kaplan can be invoked to yield a contradiction. An analogous point can be made for the so-called Diodorean tense modalities.

[2] F and P may as well be treated as "defined" predicates, i.e. as metalinguistic abbreviations.

[3] For a description of the system **Q**, see Boolos and Jeffrey [1], p. 107.

[4] See Boolos and Jeffrey [1], Chapters 17 and 18.

[5] See Prior [7], p. 176.

[6] See Prior [7], p. 176, Axiom 5.2.

[7] See Burgess [2], pp. 104–105.

[8] Actually, we could have also used different pairs of formulas, like $\alpha' \leftrightarrow G\overline{H\overline{\overline{\neg\alpha'}}}$, $\beta' \leftrightarrow H\overline{G\overline{\overline{\neg\beta'}}}$, for the same purpose.

[9] See Tarski [9].

[10] See [4] and [6].

[11] See [10].

[12] This is proved in Friedman and Sheard [3], pp. 11–13.

[13] This is proved in Friedman and Sheard [3], p. 14.

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