The influence of the Polish school in logic on mathematical philosophy*

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Abstract

This article reflects on the influence of the Polish school in logic on mathematical philosophy. The focus lies on developments after the second world war.

1 Introduction

The Polish school in logic has had a dramatic influence on philosophy in the twentieth century. The inter-war period is considered the heyday of the Polish school. And the prime example of influence of logic on mathematical philosophy that it has produced consists in Tarski's theory of truth. It has served as a paradigm of the way in which techniques from mathematical logic can fruitfully be applied to philosophical questions.

Not only Tarski's work, but the whole history of the Polish school of logic in the interwar period has in recent decades been the subject of outstanding scholarly work. Much less has been written about the influence of the Polish tradition in mathematical logic on philosophy after the second world war. My intention here is not to give an overview of the post-war history of logic in Poland and its influence on philosophy. I will merely argue that the Polish school in mathematical logic has continued to produce deep results that cry out for philosophical reflection and exploitation.

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My argument will be based on two examples with which I am familiar. I will focus on two post-war strands in mathematical logic in which the Polish school has played an important role: the theory of satisfaction classes and saturated models on the one hand, and the ultraproduct construction on the other hand. I will show that they have an impact on mathematical philosophy already, and I predict that the influence of these strands of research on mathematical philosophy will deepen in the decades to come.

2 Tarski and truth theory

Tarski played an immense role in the establishment of logic as a branch of mathematics. It is well known that Tarski did all he could to bring out the mathematical meaning of his logical results, sometimes even at the expense of the natural readability of his articles.

But Tarski was also sensitive to problems that were discussed in scientifically minded philosophical circles in Poland before the second World War. He famously gave a rigorous definition of truth in an (interpreted) formal language [Tarski 1935]. This has since then been uniformly regarded as a paradigm example of Carnap's method of logical analysis [Carnap 1950, chapter 1]. Nowadays we would regard this as a landmark accomplishment in *mathematical philosophy*.[1]

It has been observed that Tarski's primary aim in formulating his theory of truth was to make metamathematics a respectable *mathematical* enterprise [Hodges 2008]. Indeed, Tarski's work on truth can be seen as the birth of model theory as a branch of mathematical logic, which is in turn a branch of mathematics. Nonetheless, Tarski's work on truth does show that where many mathematicians consider it positively harmful for their mathematical career to engage with philosophical questions, Tarski took the opposite view. And history has shown that this was a fruitful attitude to take.

In philosophy, this has given rise to a discipline that is called *formal theories of truth*. This is a discipline that is located in philosophical logic. Nowadays it uses techniques and results of mathematical logic. But its intention is to shed light on the *philosophical* concept of truth. And, by doing so, Tarski's work has transformed the traditional philosophical debates about the nature of truth. Tarski's work has brought out the *function* that the concept of truth fulfils (it allows us to express propositions that we

[1] See [Horsten 2013], [Leitgeb 2013], [Horsten & Douven 2008].

could not express without it), and the logical laws that the concept obeys. As a side effect, it has caused philosophers to revisit and to a significant extent to doubt the alleged metaphysical content that the concept of truth possesses.[2] After Tarski's work, truth has gradually come to be seen more as a logico-linguistic concept than as a metaphysical concept.

Tarski defined what it means to be a true sentence of a language that does not itself contain that same notion of truth. After Tarski, the emphasis slowly shifted to defining notions of truth for sentences that themselves contain that very same notion of truth. The work of Kripke and of Belnap, Gupta, and Herzberger in the United States stands out here.[3]

But even today, much can be learned from Tarski's work on truth. Recall that Tarski's aim was to define truth for *mathematical* theories. His theorem on the undefinability of truth entailed that a materially adequate truth theory can only be formulated in a stronger metalanguage. Moreover, Tarski required that the metatheory must be able to carry out the logical proof techniques in model theory in which truth plays an *essential* role. This entails that the truth theory is not allowed to be *interpretable* in the background mathematical theory:[4] otherwise the background mathematical theory would be able to simulate the behaviour of the truth predicate. At the same time, model theorists tend to adopt an algebraic stance towards the mathematical theories that they study. That is, they consider all models of the mathematical theory that they investigate as being on a par. (Moreover, model theorists think of models as *first-order* models.) But this entails that a theory of truth for a mathematical theory ought to be *semantically conservative*: every model of the mathematical theory should be expandable to a model of the union of the mathematical theory and the truth theory. Otherwise the truth theory would exclude models of the background theory. In sum, Tarski's requirements entail that a truth theory should be at the same time non-interpretable and semantically conservative. Thus it is natural to wonder whether there are natural formal truth theories that meet these two requirements. This is a non-trivial question, for the requirements seem to pull in opposite directions. We will briefly return to this question later in this article.[5]

In any case, the importance of Tarski's theory of truth is not restricted to philosophy. The concept of truth has been used as the cornerstone of a highly influential theory of meaning for natural languages. The central

[2]See [Horsten 2011, chapter 2].

[3]See [Kripke 1975], [Gupta & Belnap 1993].

[4]The concept of interpretability was discovered and investigated by Tarski after the second world war.

[5]See section 3.

thesis here is that the meaning of a given sentence consists of the circumstances (or "possible worlds") in which the sentence is true. This idea was implemented in detail for fragments of English by one of Tarski's most influential students, Richard Montague [Montague 1974]. The implementation proceeded by interpreting the circumstances in which a sentence is true as *models* in the mathematical sense of the word, and by taking these models to be models of intensional higher-order logic. Thus Tarski's work on truth has had a profound influence on natural language semantics, which is a sub-field of linguistics.

Of course Tarski's influence on philosophy and related disciplines is not restricted to his work on truth. In particular, Tarski carried out groundbreaking research on the question of the demarcation of the class of logical notions [Tarski 1986]. However, I will not further comment on this line of research here.

3 Polish logic after the second world war

Even though Tarski is the most important exponent of the Polish School in logic of the inter-war period, it is well documented that the Polish School was so much more than Tarski alone. The history of the Polish School in logic up to the second world war, and its roots in developments in Polish philosophy, have been thoroughly investigated.[6] The history of logic in Poland after the second world war, and its influence on related disciplines, have as far as I know not been investigated in nearly the same level of detail. It is not my aim to write the history of logic in Poland after the second world war, and I would not be in the least qualified to do so. All I want to argue for in this article, is that it would be well worth doing so.

Just before the German invasion of Poland, Tarski left for the United States, where he founded the most influential post-war research centre in mathematical logic. And with Tarski's (and Gödel's) move to the United States, the centre of gravity of mathematical logic also moved from Europe to the United States.[7] Moreover, the main names that I associated in the previous section with Tarski's influence of mathematical logic on cognate disciplines (Kripke, Gupta, Belnap, Montague...) suggest that the centre of gravity of mathematical philosophy also moved to the United States. Thus one might think that the horrors of the second world war mark the end of the illustrious Polish school in logic and its influence on philosophy.

[6] See for instance [Wolenski 1989], [Simons 2002], [Betti 2004].

[7] Mostowski stayed in Europe, but also in set theory Tarski and his disciples set the agenda after the second world war.

In what follows, I shall argue that to think so would be to commit a grave mistake. It is true that since the second world war the United States has been a powerhouse in mathematical logic and that developments in mathematical logic that took place there have had an enormous influence on neighbouring disciplines. But the importance of Polish logicians remains enormous even after the second world war.[8]

In what follows, I want to focus on two post-war theorems in mathematical logic that have a strong Polish connection, and reflect on their possible philosophical significance. These are but examples that are intended to illustrate that the post-war Polish Logic school has produced results that simply cry out for philosophical exploitation. What follows are merely examples that I myself am somewhat familiar with; I by no means want to suggest that these examples are the most important ones!

4 Truth theory again: satisfaction classes

It is known that many axiomatic theories of truth are *proof-theoretically non-conservative*: if they are added to a background theory (say to Peano Arithmetic: PA), then they enable the combined theory to prove theorems not involving the notion of truth that cannot be proved in the background theory. A standard example is the compositional theory of truth that is called CT.[9] This theory CT consists of Peano Arithmetic with the truth predicate allowed in the induction scheme, plus the axioms that state that the truth predicate commutes with the logical connectives. CT is a very natural truth theory. But it is well-known that CT is proof-theoretically not conservative over PA. For instance, it proves the Gödel sentence for PA. Many (but not all![10]) deflationists about truth argue that since the concept of truth should be neutral about mathematical, metaphysical, physical,... matters, a good truth theory should be proof-theoretically conservative over its background theory. Thus they find CT inadequate as a truth theory for PA.

The standard argument for the proof-theoretical non-conservativeness of CT over PA uses an instance mathematical induction (in CT) in which the truth predicate occurs. Thus it is natural to consider the compositional truth theory $CT\restriction$, which is just like CT except that its induction scheme

[8]Mostowski was an immensely influential figure in the Polish school of mathematical logic after the second world war. See [Ehrenfeucht et al 2008].

[9]For a detailed discussion of CT, see [Halbach 2011, chapter 8], or [Horsten 2011, chapter 6].

[10]See [Horsten 2009].

does not contain instances in which the truth predicate occurs. It is very natural to think that there is an easy argument to show that $CT\restriction$ is proof-theoretically conservative over *PA* and thus meets the requirement that many deflationists impose on it. The argument goes as follows. Take any model of *PA*. Expand it to a model of the truth predicate by making sure that the compositional axioms hold for it —remember: we do not have to worry about the induction axiom. Then the completeness theorem for first-order logic gives the desired conclusion.[11] But this argument is fallacious, for not every *PA*-model can be expanded to a $CT\restriction$-model.

Kotlarski, Krajewski, and Lachlan in the beginning of the 1980s proved the following important theorem that bears on this question [Lachlan 1981], [Kotlarski et al 1981]:

Theorem 1 *A countable model is recursively saturated if and only if it has a satisfaction class.*

Being recursively saturated means that all recursive types are realised, and having a satisfaction class means that it can be extended with a truth predicate for which the compositional truth axioms hold. The left-to-right direction of Theorem 1 is due to Lachlan; the right-to-left direction is due to Kotlarski, Krajewski, and Lachlan. Lachlan is a Canadian logician; Kotlarski and Krajewski are Polish.[12] The proof of Theorem 1 is too complicated to figure in what is taught in a typical intermediate logic course.

Thus the Kotlarski-Krajewski-Lachlan theorem teaches us that things are not as simple as they appear at first sight. Nonetheless, one direction of Theorem 1 (the direction that is due to Kotlarski et al) can be used to show that $CT\restriction$ is proof-theoretically conservative over *PA* after all. This is because by a result of Barwise and Schlipf,[13] every countable model of *PA* has an elementary equivalent recursively saturated extension. So for some deflationists, $CT\restriction$ can be a satisfactory truth theory for *PA* after all: it depends on whether they cash out the requirement of 'theoretical neutrality' on the truth predicate in a semantical or in a proof theoretical way. By thus settling the question of the conservativeness of $CT\restriction$, and (perhaps more importantly) by extensionally distinguishing the concepts of proof theoretic and semantic conservativeness, Kotlarski, Krajewski, and Lachlan have made a signal contribution to mathematical philosophy. They give us an

[11] Even some of the strongest mathematical logicians of the twentieth century have fallen for this: see for instance [Feferman 1991, Lemma 2.4.2].

[12] Kotlarski died some years ago: see *In Memoriam: Henryk Kotlarski* (Bulletin of Symbolic Logic **16**(2010), p. 145).

[13] See [Barwise & Schlipf 1976].

analysis that can be seen as a refinement and deepening of Tarski's analysis of the concept of truth.

Now let us return to the Tarskian requirement of a truth theory S for a given mathematical theory that was described in section 2 (and let the background mathematical theory be *PA*): S should be semantically conservative over *PA*. Again, the theory $CT\upharpoonright$ initially looks like a good candidate. $CT\upharpoonright$ is a natural truth theoryThe standard argument for showing that CT is not conservative over *PA* cannot be carried out in $CT\upharpoonright$. But the Kotlarski-Krajewski-Lachlan theorem shows that despite this, $CT\upharpoonright$ does not satisfy requirement (2). Although $CT\upharpoonright$ is proof-theoretically conservative over *PA*, it is not *semantically* conservative over *PA*, for not all *PA*-models are recursively saturated.[14]

Of course this is just the beginning of the story of the Tarskian challenge. The foregoing considerations still leave the question whether there is an attractive truth theory that meets (1) and (2) wide open. It merely indicates that the problem is not as simple as one might at first think. I will not pursue this story further here, save to say that there is a natural truth theory, due to Martin Fischer, that meets both Tarskian demands. For an extended philosophical discussion of Tarski's requirements and the way in which they are met by Fischer's theory, see [Fischer & Horsten unpubl].

Note, finally, that the considerations in this section constitute merely the beginning of the use of the theory of satisfaction classes and saturated models in the theory of truth. Indeed, the foregoing suggests that more insight into the concept of truth that may be obtained by investigating the concepts of recursive saturation and of satisfaction class.

5 Infinitesimal probabilities

A second important mathematical theorem that has Polish post-war roots is Łoś' celebrated theorem [Łoś 1955]:

Theorem 2 *Let $\{M_i \mid i \in I\}$ be a set of models indexed by a set I, and let F be an ultrafilter on I. Then any first-order sentence ϕ is true in the ultraproduct relative to F of the set $\{M_i \mid i \in I\}$ if and only if $\{i \mid M_i \models \phi\} \in F$.*

The ultraproduct model thus "inherits" first-order properties from its factors. For instance, if all its factors are models of the field axioms, then the

[14]Indeed, more is true. Not even the theory TB, which consists of *PA* (with truth allowed in the induction scheme) plus the *Tarski-Biconditional sentences* of the form $T(\overline{g(A)}) \leftrightarrow A$, where A ranges over arithmetical sentences and g is the function that assigns Gödel codes to sentences, is semantically conservative over *PA*. See [Strollo 201?].

ultraproduct model will also make the field axioms true. Thus, on account of Łoś' theorem, ultrapower models are suitable for constructing for given theories: models that have special properties.

In set theory, the ultraproduct construction has for decades been a workhorse for constructing models of the standard axioms of set theory (and of large cardinal axioms). And also in analysis, ultraproduct constructions have been used to generate non-standard models of the principles of analysis. In particular, ultraproduct constructions can be used to construct models of analysis that contain infinitely small real numbers (*infinitesimals*), thus proving Leibniz's interpretation of analysis to be coherent after all, centuries after the time when this interpretation was first proposed. These models are investigated in *non-standard analysis* [Robinson 1961].

Non-standard analysis has never become very popular with mathematical analysts even though it arguably yields more intuitive proofs of basic theorems than the standard $\epsilon - \delta$ approach does. The reason for this is the so-called *transfer phenomenon*, which says that a first-order theorem is true in a non-standard model if and only if it is true in the standard model. Thus, in some sense, non-standard models do not yield anything new. So analysts don't feel that they need them.

But for probability theory, which is of course intimately related to analysis, the situation is somewhat different. For there are basic chance situations that standard (or classical) probability cannot model, but that can be modelled by certain natural non-standard probability functions that are generated by an ultraproduct construction.

Classical probability, as encapsulated in the Kolmogorov axioms [Kolmogorov 1933],[15] is notoriously bad at describing uniform distributions (*fair lotteries*) on infinite sample spaces. A uniform probability distribution on a countably infinite space is simply incompatible with the Kolmogorov axioms. And the only uniform probability distribution on uncountable spaces that is compatible with the Kolmogorov axioms is the uniform zero distribution. But this uniform zero distribution effaces the distinction between impossibilities and infinitely remote contingencies: both are assigned probability zero.

Using the ultraproduct construction, non-standard probability functions that do model fair lotteries on countably infinite and uncountably infinite sample spaces can be defined [Wenmackers & Horsten 2013], [Benci et al 2013]. Such functions assign a non-zero but infinitesimal probability to each point event, and probabilities of the point events 'sum' to 1. Łoś' theorem guarantees that these nonstandard probability functions satisfy the basic Kol-

[15] The Kolmogorov axioms include the principle of σ-additivity.

mogorov axioms.[16] Moreover, for every sample space, these functions are defined on the full power set, so that no sets are classified as non-measurable.

In recent years, a priori philosophical arguments have been advanced that intend to show that infinitesimals cannot play a useful role in probability theory.[17] So there is a puzzle. Either the generalised probability functions that are produced by the ultraproduct construction have properties that are philosophically abhorrent, or the philosophical arguments against infinitesimal probabilities are fallacious. Again, this is not the place to pursue this question further.[18] It suffices to say that here we have another example of a way in which methods from mathematical logic inform mathematical philosophy.

The mathematical properties of the infinitesimal probability functions defined in [Benci et al unpubl] are presently only very incompletely understood. And we know from other areas of mathematical logic (set theory, model theory) that ultraproduct constructions are extremely versatile and powerful, and can yield deep insight. So there is every reason to believe that in the theory of infinitesimal probabilities, we a present have a poor insight into the possibilities and impossibilities. We have only stratched the surface: there must be much more to come.

6 Conclusion

I have considered two developments in mathematical logic after the second world war in which Polish logicians played a key role. The first development is of somewhat more recent date (the early 1980s), but the second development took place more than fifty years ago. So one might wonder why they have not influenced mathematical philosophy before now. Wolenski suggests that during the communist period in Poland after the second world war it was ideologically unacceptable to engage in (mathematical) philosophy to the extent that it had been done before the war.[19] This is must certainly be part of the explanation. But it is also important to realise that the relevant results and proof techniques are substantially more difficult than the work in mathematical logic that has been exploited

[16] Instead of σ-additivity, these probability functions satisfy a generalised form of infinite additivity.

[17] See for instance [Williamson 2007], [Easwaran unpubl], [Pruss 2012].

[18] Many of the objections against infinitesimal probabilities are addressed in [Benci et al unpubl].

[19] See [Wolenski 2013].

by mathematical philosophers until recently. It just takes time for philosophers to digest the philosophical relevance of deep results in mathematical logic.

At any rate, I hope that the foregoing has shown that the Polish school in logic remained very strong after the second world war, and that we may expect its results deeply to impact on developments in mathematical philosophy for decades to come.

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