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# An Axiomatic Investigation of Provability as a Primitive Predicate

## 1 Introduction

The intuitive or informal notion of provability, if it is a coherent notion at all, appears to express a property that holds of certain objects (propositions perhaps, or sentences, or statements) and fails to hold of others. It therefore appears that in an axiomatic investigation of this notion, it ought to be treated as a *predicate*. Nevertheless, the majority of axiomatic investigations of intuitive provability formalize this notion as a *sentential operator*. This has the well-known disadvantage that it then becomes impossible to quantify over the objects that are informally provable. For instance, on this approach it is impossible to express in the formal language:

> Some propositions (or sentences, or statements) are intuitively provable.

The reason for formalizing intuitive provability as a sentential operator is that, like the notion of truth and the notion of necessity, it is infected with paradox. Tarski showed that in the context of even a weak theory of arithmetic, the naïve axiomatization of the notion of truth leads to a contradiction. Kaplan and Montague (1960) have shown that in the context of a sufficiently strong arithmetical background theory, apparently sound principles governing a knowability predicate jointly imply a contradiction. A few years later, Montague (1963) showed that the argument of the Paradox of the Knower can also be used to generate a paradox about the notion of necessity. Thomason (1980) then used a different but closely related argument

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to bolster the claim that the notion of rational belief is also paradoxical. In recent years, stronger inconsistency and $\omega$-inconsistency results have been obtained for the treatment of truth,[2] intuitive provability and necessity as a predicate. Moreover, all these inconsistency results are deeply related. They all fall under the category of *diagonal arguments*.

In sum, a good deal is currently known about which intuitively plausible principles concerning intuitive provability (or necessity, or rational belief) are jointly inconsistent or $\omega$-inconsistent. In the case of truth, this has not prevented logicians from formulating interesting and well-motivated axiomatizations of truth as a predicate. Here the Kripke–Feferman theory KF and Cantini's system VF come foremost to mind. We have at present nothing comparable for intuitive provability, necessity and rational belief. Very little is known, for example, about which plausible principles concerning intuitive provability *can* be consistently combined to form a unified system. An explanation for this situation is not hard to find. There exist illuminating *semantical* theories of truth – such as Kripke's treatment of truth as a partial predicate. These semantical theories give us guidance in our attempts at formulating interesting axiomatic theories of truth. The semantical constructions can be naturally, albeit only partially, described in axiomatic theories. There exists at present nothing like this for the notions of intuitive provability, necessity and rational belief.

In the present article, we concentrate *mainly* on the informal notion of provability, treated as a predicate. In the next section, a provisional diagnosis of the Paradox of the Knower is presented. This provisional diagnosis motivates a specific strategy for constructing axiomatic treatments of informal provability that has been pursued to some extent in the literature. In Section 3 axiomatic systems that have been constructed along these lines are revisited and extended, and some of the metamathematical properties of these systems are investigated. These metamathematical investigations are to some extent guided by questions that arose in the literature on *Epistemic Arithmetic*,[3] where the notion of informal provability is treated not as a predicate but as an *operator*. In the course of our metamathematical investigations it will emerge that the provisional diagnosis of the Paradox of the Knower is in the end untenable. In Section 4, a philosophical investigation will be made into the conceptual connection between the notions that are affected by Liar-like paradoxes on the one hand (such as necessity, informal provability, and perhaps rational belief), and the notion of truth on the other hand.

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[2]See, for instance, McGee 1985 and Friedman and Sheard 1987.

[3]The classical reference here is Shapiro 1985b.

## 2 Provisional operational diagnosis of the Paradox of the Knower

Define the language $\mathcal{L}_{\mathsf{PEA}}$ of *Predicate Epistemic Arithmetic* as the language of first-order arithmetic (including names for all primitive recursive functions) plus a new first-order predicate P. The *Paradox of the Knower* follows in the context of a weak theory of arithmetic (such as Robinson's arithmetic Q, formulated in the extended language $\mathcal{L}_{\mathsf{PEA}}$) plus an epistemic principle called *Reflexivity*:

$$\mathrm{P}\ulcorner \underline{A} \urcorner \rightarrow A \quad \text{for all formulas } A \in \mathcal{L}_{\mathsf{PEA}},$$

and the inference rule of *Provabilitation*:

$$A \Rightarrow \mathrm{P}\ulcorner \underline{A} \urcorner.$$

The resulting (inconsistent) system is a subsystem of Feys' system T of modal logic, except that it is formulated in the context of an arithmetic theory, and that the intensional notion in question is treated as a predicate.

Each of these two principles concerning informal provability strongly appears to be *sound* – and yet jointly they are inconsistent. Suppose we want to adopt a principle of minimal mutilation and plan to weaken, if possible, *just one* of these principles. Let us briefly review the options:

### *Reflexivity*

On the one hand, Thomason (1980, 392–93) showed that even if we weaken Reflexivity to $\mathrm{P}\ulcorner \underline{\mathrm{P}\ulcorner \underline{A} \urcorner \rightarrow A} \urcorner$, then, given a few other seemingly weak epistemic principles, a contradiction still follows. Moreover, even if we restrict Reflexivity to formulas *A in which* P *does not occur*, then, in the presence of some seemingly weak epistemic principles, a contradiction follows.[4]

On the other hand, if we leave out Reflexivity completely, then the resulting system is a subsystem of the predicate version of the *Löb system* GL. The intuitive provability predicate can then be read as provability in a formal system. The distinctness of the notion of intuitive provability from provability in a formal system is then no longer reflected in the axiomatization. Also if we restrict Reflexivity to formulas $A$ in which P does not occur at all, P can be read as provability in a formal system. This seems unattractive.

[4]We may assume that P satisfies Provabilitation. If P also distributes over "→", and the principle S4 holds for P, then P is a *provability predicate* (in the sense of Boolos and Jeffrey 1989, 185). Therefore Löb's theorem would hold. But if the epistemic theory also contains Reflexivity for *non-epistemic* formulas, then it proves $\mathrm{P}\ulcorner \underline{0=1} \urcorner \rightarrow (0=1)$. Then by Löb's theorem for this system, $0=1$ is provable.

At the same time, Reflexivity is very basic. The fact that proof entails truth follows from the *meaning* of the concept of proof.

*Provabilitation*

The motivation of Provabilitation is somewhat less immediate than that of Reflexivity. Provabilitation is motivated by an inductive argument: induction on the number of occurrences of Provabilitation in a proof of the formal system.

It was mentioned above that if Provabilitation is replaced by its consequence $P\ulcorner P\ulcorner \underline{A}\urcorner \rightarrow A\urcorner$, then, given a few other seemingly exceedingly weak epistemic principles, a contradiction still follows. This suggests one "minimal" way to avoid the paradox:

*Don't provabilitate Reflexivity!*

If Provabilitation of Reflexivity somehow really is at the root of the Paradox of the Knower, then we might hope that many (perhaps all) other plausible-looking epistemic principles (such as the principle S4, for example) can consistently be provabilitated. In other words, everything that *appears* informally provable really *is* informally provable, *except* that informal provability entails truth.

Of course, even if this would be so, we would still be in a *philosophically* unsatisfying situation. For by itself, this strategy can only give us a deeper *technical* understanding of the Paradox of the Knower. It does not give us *conceptual* understanding. It does not answer the philosophical question: *why* is it that Reflexivity cannot consistently be provabilitated?

In any case, the above heuristic slogan is implicitly or explicitly embraced by several authors in the literature. Most of the consistent systems presented by Friedman and Sheard (1987), by Germano (1970) and by Niebergall (1991) implicitly incorporate this proposal.

# 3 Predicate Epistemic Arithmetic

In this section we will take the tentative diagnosis of the Paradox of the Knower in the previous section as our guiding heuristic principle for constructing systems of Predicate Epistemic Arithmetic. Also, some of the metatheoretic properties of these systems will be investigated.

## 3.1 A basic formal system of Predicate Epistemic Arithmetic

Axiomatic systems have been proposed which are perhaps most naturally interpreted as axiomatizations of informal provability as a primitive predicate – although in the literature it is often explicitly left open whether the primitive predicate is to be interpreted as truth, necessity, provability or some other notion. Specifically, Germano, Friedman and Sheard, and Niebergall all consider slight variations on the system which I will call $\mathsf{PEA}^0$, and which is defined as consisting of [5]

1. PA in the extended language $\mathcal{L}_{\mathsf{PEA}}$, where occurrences of P are allowed in instances of the induction scheme;

2. $\mathrm{P}\ulcorner\underline{A}\urcorner \rightarrow (\mathrm{P}\ulcorner\underline{A \rightarrow B}\urcorner \rightarrow \mathrm{P}\ulcorner\underline{B}\urcorner)$ for all formulas $A, B \in \mathcal{L}_{\mathsf{PEA}}$;

3. $\mathrm{P}\ulcorner\underline{A}\urcorner \rightarrow \mathrm{P}\ulcorner\underline{\mathrm{P}\ulcorner\underline{A}\urcorner}\urcorner$ for all formulas $A \in \mathcal{L}_{\mathsf{PEA}}$;

4. $\mathrm{Bew}_{\mathsf{PA}}(\ulcorner\underline{A}\urcorner) \rightarrow \mathrm{P}\ulcorner\underline{A}\urcorner$ for all formulas $A \in \mathcal{L}_{\mathsf{PEA}}$, where $\mathrm{Bew}_{\mathsf{PA}}$ is the standard provability predicate for PA in the extended language;

5. $\mathrm{P}\ulcorner\underline{A}\urcorner \rightarrow A$ for all formulas $A \in \mathcal{L}_{\mathsf{PEA}}$.

These authors show that $\mathsf{PEA}^0$ is consistent. Moreover, Friedman and Sheard (1988) show that a system which is somewhat stronger than $\mathsf{PEA}^0$ has epistemic counterparts of the disjunction and numerical existence properties which hold for systems of constructivistic arithmetic.[6] Their proofs carry over to $\mathsf{PEA}^0$:

(ENEP) if $\mathsf{PEA}^0 \vdash \mathrm{P}\ulcorner\underline{A}\urcorner \vee \mathrm{P}\ulcorner\underline{B}\urcorner$ then $\mathsf{PEA}^0 \vdash \mathrm{P}\ulcorner\underline{A}\urcorner$ or $\mathsf{PEA}^0 \vdash \mathrm{P}\ulcorner\underline{B}\urcorner$, for all sentences $A, B \in \mathcal{L}_{\mathsf{PEA}}$;

(EDP) for all formulas $A \in \mathcal{L}_{\mathsf{PEA}}$, if $\mathsf{PEA}^0 \vdash \exists x\, \mathrm{P}\ulcorner\underline{A}\urcorner$ then, for some $n \in \mathbb{N}$, $\mathsf{PEA}^0 \vdash \mathrm{P}\ulcorner\underline{A(\underline{n}/x)}\urcorner$.

The basic system that is taken as a starting point in the present paper is called PEA. It is somewhat stronger than $\mathsf{PEA}^0$, but very similar to it. PEA consists of

[5] See Germano 1970, 36–37; Friedman and Sheard 1987, 7, chart 1; Niebergall 1991, 36. $\mathsf{PEA}^0$ is considerably stronger than the system that was proposed by Myhill (1960, 469–70), which appears to be one of the earliest attempts to consistently formalize informal provability as a primitive predicate. It is worth mentioning that Myhill in this paper formulates a version of the Paradox of the Knower (pp. 469–70), and is therefore perhaps entitled to credit for discovering the paradox around the same time as Kaplan and Montague did.

[6] See, e.g., Troelstra and van Dalen 1988, 1:139.

**Axiom 1** PA in the extended language $\mathcal{L}_{PEA}$, where occurrences of P are allowed in instances of the induction scheme;

**Axiom 2** $P\ulcorner A \urcorner \rightarrow A$ for all formulas $A \in \mathcal{L}_{PEA}$;

**Axiom 3** $Bew_{BPEA}(\ulcorner A \urcorner) \rightarrow P\ulcorner A \urcorner$ for all formulas $A \in \mathcal{L}_{PEA}$.

Here $Bew_{BPEA}$ is the standard provability predicate for the *basis* of PEA, which is the theory consisting of

**Basis 1** PA in the extended language $\mathcal{L}_{PEA}$, where occurrences of P are allowed in instances of the induction scheme;

**Basis 2** $P\ulcorner A \urcorner \rightarrow (P\ulcorner A \rightarrow B \urcorner \rightarrow P\ulcorner B \urcorner)$ for all formulas $A, B \in \mathcal{L}_{PEA}$;

**Basis 3** $P\ulcorner A \urcorner \rightarrow P\ulcorner P\ulcorner A \urcorner \urcorner$ for all formulas $A \in \mathcal{L}_{PEA}$.

In this system, Axiom 3 functions as a weakened version of the Provabilitation rule. It respects the injunction of the previous section not to provabilitate Reflexivity. Otherwise it is kept as strong as possible. Basis 2 is called the *Distributivity axiom*. Basis 3 is called the 4-*axiom*.

PEA is the most straightforward implementation of the strategy outlined in the previous section. The first system that comes to mind when one wants to formalize the notion of informal provability is the modal system S4. The Paradox of the Knower teaches us that if informal provability is treated as a *predicate* in the formalization, then the resulting system is inconsistent. If this inconsistent system is then slightly weakened by disallowing Reflexivity to be provabilitated, what is obtained is precisely the system PEA.

## 3.2 Candidate additional axioms

Let us now briefly consider some principles which can be used to strengthen the basic system PEA. These putative new axioms take the form of iteration principles for P, principles that govern the interaction between iteration of P and negation, and principles that govern the interaction between P and quantification. Counterparts of these principles have been investigated in (operator) modal propositional and predicate logic and in the literature on Epistemic Arithmetic. But these principles remained largely outside of the scope of the investigations of Friedman and Sheard (1987, 1988) and of Niebergall (1991).

First, there is the converse 4:

$$P\ulcorner P\ulcorner A \urcorner \urcorner \rightarrow P\ulcorner A \urcorner,$$

which is called C4. Of course C4 is *provable* in PEA. But we can contemplate whether it is consistent to include C4 even in the *basis* of PEA.

For a second candidate additional axiom for PEA, consider the following principle of propositional (operator) modal logic, called *Fitch's Axiom*:

$$\Box\neg\Box A \rightarrow \Box\neg A.$$

It is well known that in all extensions of the system T of propositional modal logic, this principle cannot be added without trivializing the modal operator. This observation is due to Fitch (1963). But his argument depends crucially on an application of the Provabilitation rule to a sentence obtained by an application of the Reflexivity axiom. The corresponding argument in PEA would therefore break down: we would not be allowed to provabilitate. Therefore the possibility arises that the "predicate counterpart" F:

$$P\ulcorner\underline{\neg P\ulcorner\underline{A}\urcorner}\urcorner \rightarrow P\ulcorner\underline{\neg A}\urcorner$$

of Fitch's Axiom can be added to PEA without trivializing the informal provability predicate. In the context of PEA, F implies the predicate counterpart of the S4.1-axiom M (also known as McKinsey's Axiom):

$$P\ulcorner\underline{\neg P\ulcorner\underline{\neg A}\urcorner}\urcorner \rightarrow \neg P\ulcorner\underline{\neg P\ulcorner\underline{A}\urcorner}\urcorner,$$

i.e., it is easily shown that

**Proposition 1** PEA+F $\vdash$ M.

I have argued elsewhere that the operator version of M, and variants of it, have some degree of plausibility.[7]

Thirdly, the converse of F (CF) can be considered:

$$P\ulcorner\underline{\neg A}\urcorner \rightarrow P\ulcorner\underline{\neg P\ulcorner\underline{A}\urcorner}\urcorner.$$

In all extensions of the operator modal logic S4, (the operator part of) CF is derivable. But again this derivation makes use of the ability to necessitate sentences obtained by Reflexivity, the counterpart of which we do not have in PEA.

The last putative additional axiom that we will consider is motivated from the literature on Epistemic Arithmetic.[8] There it was noted that it is possible to formulate epistemic analogues of *Church's Thesis*. In $\mathcal{L}_{\mathsf{PEA}}$, the counterpart

[7] See Horsten 1997.

[8] See Shapiro 1985a, 30–31.

of this epistemic principle for informal provability treated as a predicate can be expressed as follows:

$$\mathrm{P}\ulcorner\forall x\,\exists y\,\mathrm{P}\ulcorner\underline{A(\dot{x},\dot{y})}\urcorner\urcorner \rightarrow \exists e\,\forall x\,\exists y\,[T(e,x,y)\wedge A(x,U(y))],$$

where $T$ is Kleene's $T$-predicate and $U$ is the $U$-function symbol. This principle is called ECT (*Epistemic Church's Thesis*). The question arises whether ECT can be consistently added to PEA. Kreisel once stated (1987, 86–87):

> *Truth and general* [i.e., informal] *provability*; at least so far, a distinction without much difference. ... nothing is explicitly formulated about general provability that does not also hold for truth ...

In other words, Kreisel is saying that, as far as we know, the theory of informal provability is simply a subtheory of the theory of truth. ECT may be a candidate counterexample to Kreisel's statement – the *only* candidate counterexample to date, as far as I can tell.

## 3.3 Elementary observations about PEA and related systems

*Two ways of strengthening* PEA

We will construct variants of PEA in the following two ways:

- By adding a principle $I$ to PEA, yielding a stronger system PEA $+ I$;
- By strengthening the basis of PEA, and modifying Axiom 3 accordingly. For a given principle $S$, Axiom 3 is replaced by

  $$\mathrm{Bew}_{\mathsf{BPEA}+S}(\ulcorner\underline{A}\urcorner) \rightarrow \mathrm{P}\ulcorner\underline{A}\urcorner.$$

  The resulting system is then called PEA $+ S^{\mathrm{i}}$.[9]

In this way we obtain PEA $+$ $\mathsf{M}^{\mathrm{i}}$, PEA $+$ ECT, etc. We can of course also formulate "hybrid" systems, such as PEA $+$ $\mathsf{F}^{\mathrm{i}}$ $+$ ECT.

*Consistency results*

We now present some consistency results and computations of arithmetical strength.

**Theorem 2** PEA $+$ $\mathsf{CF}^{\mathrm{i}}$ $+$ F $+$ C4 *has an* $\omega$*-model.*

[9] The superscript "i" indicates that $S$ is added to the "inner logic" of the system. More on this below.

The proof of this theorem is somewhat involved and is relegated to the appendix. This theorem entails of course:

**Corollary 3** PEA *has an $\omega$-model.*

We recall the definition of the notion of *uniform reflection* for PA:

**Definition 4** U-Rfn(PA) *is the scheme*

$$\mathrm{Bew}_{\mathsf{PA}}(\ulcorner\underline{A}\urcorner) \rightarrow A \quad \textit{for all formulas } A \in \mathcal{L}_{\mathsf{PA}}.$$

Then we have:

**Proposition 5** *For all formulas $A \in \mathcal{L}_{\mathsf{PEA}}$:*

$$\mathsf{PEA} + \mathsf{F}^{\mathsf{i}} \vdash A \quad \Leftrightarrow \quad \mathsf{PA} + \text{U-Rfn}(\mathsf{PA}) \vdash A.$$

*Proof:*

1. It is immediate that PA + U-Rfn(PA) is arithmetically conservative over $\mathsf{PEA} + \mathsf{F}^{\mathsf{i}}$.
2. It remains to be shown that $\mathsf{PEA} + \mathsf{F}^{\mathsf{i}}$ is arithmetically conservative over PA + U-Rfn(PA). Consider the translation $\tau$, which is such that
   a) $\tau(A{*}B) = \tau(A) * \tau(B)$ for $*$ any binary logical connective;
   b) $\tau(*A) = *\tau(A)$ for $*$ any unary logical connective;
   c) $\tau(\mathrm{P}t) = \mathrm{Bew}_{\mathsf{PA}}(t)$ if $t \neq \ulcorner\underline{A}\urcorner$ for some formula $A$; $\tau(\mathrm{P}(\ulcorner\underline{A}\urcorner)) = \mathrm{Bew}_{\mathsf{PA}}(\ulcorner\underline{\tau(A)}\urcorner)$ otherwise.
   d) if $t = \ulcorner\underline{A}\urcorner$ for some formula $A$ and $s \neq \ulcorner\underline{B}\urcorner$ for every formula $B$, then $\tau(t{=}s) = (\ulcorner\underline{\tau A}\urcorner = s)$; if $t, s \neq \ulcorner\underline{A}\urcorner$ for every formula $A$, then $\tau(t{=}s) = (t{=}s)$; etc.

   It then suffices to verify by induction on length of proofs that $\tau$ translates proofs in $\mathsf{PEA} + \mathsf{F}^{\mathsf{i}}$ into proofs in PA + U-Rfn(PA).

■

When combined with Proposition 1, this last proposition implies that $\mathsf{PEA} + \mathsf{M}^{\mathsf{i}}$ is consistent.

In view of these results, one might suspect that as long as we do not provabilitate Reflexivity, we can *have it all*, i.e., we can consistently add F, CF, 4 and C4 simultaneously to the basis of PEA. However, already $\mathsf{PEA} + \mathsf{C4}^{\mathsf{i}}$ is inconsistent. To see this, consider the theory S, which is defined as the first-order closure of

- PA in the extended language $\mathcal{L}_{\mathsf{PEA}}$, with occurrences of P allowed in the induction scheme;
- $P\ulcorner \underline{A} \urcorner \rightarrow (P\ulcorner \underline{A \rightarrow B} \urcorner \rightarrow P\ulcorner \underline{B} \urcorner)$;
- $P\ulcorner \underline{A} \urcorner \leftrightarrow P\ulcorner \underline{P\ulcorner \underline{A} \urcorner} \urcorner$;
- the inference rules $A \Rightarrow P\ulcorner \underline{A} \urcorner$ (Provabilitation) and $P\ulcorner \underline{A} \urcorner \Rightarrow A$ (Co-Provabilitation).

**Theorem 6 (Friedman and Sheard)** S *is inconsistent.*

On the basis of the inconsistency of S, the inconsistency of $\mathsf{PEA} + \mathsf{C4}^{\mathrm{i}}$ can be shown:

**Lemma 7** *For all* $A \in \mathcal{L}_{\mathsf{PEA}}$*:* $\mathsf{S} \vdash A \;\Rightarrow\; \mathsf{PEA}+\mathsf{C4}^{\mathrm{i}} \vdash P\ulcorner \underline{A} \urcorner$.

*Proof:* Induction on the length of derivations in S. ■

**Theorem 8 (Leitgeb)** $\mathsf{PEA} + \mathsf{C4}^{\mathrm{i}}$ *is inconsistent.*

*Proof:* By Friedman and Sheard's theorem, $\mathsf{S} \vdash \bot$. So by the previous lemma, $\mathsf{PEA} + \mathsf{C4}^{\mathrm{i}} \vdash P\ulcorner \underline{\bot} \urcorner$. An application of the Reflexivity axiom then gives us an outright contradiction. ■

*Inner versus outer logic*

There is an analogy between certain axiomatic theories of truth and the systems of Predicate Epistemic Arithmetic that we are considering. Reinhardt has noted that the system KF is only *partially* sound for the notion of truth. Let $L$ be the Liar sentence. Then KF proves *both* $L$ and $\neg T\ulcorner \underline{L} \urcorner$. In other words, some sentences that are proved by KF are *denied truth* by KF itself. So we cannot believe everything that KF proves.[10]

A similar phenomenon occurs for PEA and its relatives. Consider the absolute Gödel sentence $G$, i.e., the sentence $G$ such that $\mathsf{PA} \vdash G \leftrightarrow \neg P\ulcorner \underline{G} \urcorner$. In the "weak" system $\mathsf{PEA}^0$, $G$ is provable. But at the same time, $\mathsf{PEA}^0$ denies $G$ to be informally provable. In other words, $\mathsf{PEA}^0$ explicitly denies that all its derivations are "honest to God" proofs.

Reinhardt's proposed solution to this problem was to restrict the attention to the "inner logic" of the respective systems. For instance, for KF, we should restrict our attention to the (in principle axiomatizable) subsystem consisting

[10] Something similar holds for Cantini's VF.

of those sentences $A$ such that $\mathsf{KF} \vdash \mathrm{T}\ulcorner\underline{A}\urcorner$. In a similar vein, presumably, in the case of PEA we should restrict our attention to the sentences $A$ such that $\mathsf{PEA} \vdash \mathrm{P}\ulcorner\underline{A}\urcorner$.

Let IPEA designate the inner logic for PEA, etc. Here are the formalizations of the inner logics of some of the variants of PEA that we have been considering:

- $\mathsf{IPEA}^0 \equiv \mathsf{PA} + (A \Rightarrow \mathrm{P}\ulcorner\underline{A}\urcorner)$;
- $\mathsf{I}(\mathsf{PEA}+\mathsf{F}^{\mathsf{i}}) \equiv \mathsf{IPEA} + (\mathrm{P}\ulcorner\underline{\mathrm{P}\ulcorner\underline{A}\urcorner}\urcorner \rightarrow \mathrm{P}\ulcorner\underline{A}\urcorner)$;
- $\mathsf{I}(\mathsf{PEA}+\mathsf{F}^{\mathsf{i}}+\mathsf{C4}^{\mathsf{i}}) \equiv \mathsf{IPEA} + (\mathrm{P}\ulcorner\underline{\mathrm{P}\ulcorner\underline{A}\urcorner}\urcorner \rightarrow \mathrm{P}\ulcorner\underline{A}\urcorner) + (\mathrm{P}\ulcorner\underline{\neg\mathrm{P}\ulcorner\underline{A}\urcorner}\urcorner \rightarrow \mathrm{P}\ulcorner\underline{\neg A}\urcorner)$.

Whereas $\mathsf{PEA}^0$, PEA, … cannot be regarded as *sound* systems for intuitive provability, $\mathsf{IPEA}^0$, IPEA, … *are* sound formalizations of the intuitive notion of provability. It is hardly necessary to mention that these inner logics are *significantly weaker* than the "outer logics" from which they are derived.

But this means that the situation is quite different from the way it was made to appear in Section 2. There it was suggested that essentially the only price that needs to be payed for avoiding the Paradox of the Knower is to refrain from provabilitating Reflexivity. Now we see that even if Reflexivity is only assumed *as a hypothesis*, so to speak, pathological results follow. When we focus on the inner logic of PEA, we see that Reflexivity is *not* contained in it – but it *does* contain full Provabilitation! So the blame for the Paradox of the Knower is now shifted from Provabilitation to Reflexivity. We have given up our faith in full Reflexivity altogether.

Arrived at this point, one may start to wonder why we even bother to investigate these outer logics. There is at least one good reason. If we find an $\omega$-model $\mathcal{I}$ for an outer logic containing Reflexivity, then we immediately have a *sound* model for the associated inner logic, in the precise sense that for all $A$, if $\ulcorner A\urcorner \in \mathcal{I}(\mathrm{P})$, then $\mathcal{I} \models A$. There may be additional reasons. Perhaps an argument can somehow be given that we are entitled to a *larger* part of the outer logic than what is contained in the corresponding inner logic?

### *Relations with intuitionistic arithmetic*

It is natural to ask whether systems of Predicate Epistemic Arithmetic have the epistemic counterparts of the intuitionistic disjunction property and numerical existence property. It was mentioned in Section 3.1 that $\mathsf{PEA}^0$ has these properties. But also stronger systems of Predicate Epistemic Arithmetic have them. For instance, we have:

**Proposition 9** PEA + $\mathsf{F}^{\mathsf{i}}$ *has EDP and ENEP.*

*Proof:* By Friedman and Sheard 1988, EDP and ENEP are equivalent. So it suffices to show that PEA + $\mathsf{F}^{\mathsf{i}}$ has ENEP.

Consider the following model $\mathcal{M}$, based on the natural number structure. $\mathcal{M}$ is determined by what it assigns to the provability predicate P. We let the extension of P be the theory $K$, which is defined as

$$\mathsf{PA} + 4 + \mathsf{F} + (A \Rightarrow \mathrm{P}\ulcorner \underline{A} \urcorner) + (\neg A \Rightarrow \neg \mathrm{P}\ulcorner \underline{A} \urcorner).$$

It is easily verified that $\mathcal{M}$, thus defined, is a model for PEA + $\mathsf{F}^{\mathsf{i}}$. Next we show, by an induction on the length of proofs in $K$, that for all $A$, if $K \vdash A$, then PEA+$\mathsf{F}^{\mathsf{i}} \vdash \mathrm{P}\ulcorner \underline{A} \urcorner$.

Now suppose $K \vdash \exists x\, \mathrm{P}\ulcorner \underline{A} \urcorner$. Then $\mathcal{M} \models \exists x\, \mathrm{P}\ulcorner \underline{A} \urcorner$. So $\mathcal{M} \models \mathrm{P}\ulcorner \underline{A(\underline{n}/x)} \urcorner$ for some natural number $n$. Therefore $\ulcorner \underline{A(\underline{n}/x)} \urcorner$ belongs to the extension of P, i.e., $K \vdash A(\underline{n}/x)$. Therefore PEA+$\mathsf{F}^{\mathsf{i}} \vdash \mathrm{P}\ulcorner \underline{A(\underline{n}/x)} \urcorner$. ■

These properties can be taken to be about the *inner* logic of the respective systems.

In fact, the proofs of these propositions yield more information. They show that ECT is consistent with the relevant systems of Predicate Epistemic Arithmetic:

**Proposition 10** ECT *is consistent with* PEA + $\mathsf{F}^{\mathsf{i}}$.

*Proof:* Take the model $\mathcal{M}$ from the proof of the previous proposition. We will show that ECT holds in $\mathcal{M}$. Suppose $\mathcal{M} \models \mathrm{P}\ulcorner \underline{\forall x\, \exists y\, \mathrm{P}\ulcorner \underline{A(\dot{x}, \dot{y})} \urcorner} \urcorner$. Then $K \vdash \forall x\, \exists y\, \mathrm{P}\ulcorner \underline{A(\dot{x}, \dot{y})} \urcorner$, whereby

$$\mathsf{PEA} + \mathsf{F}^{\mathsf{i}} \vdash \mathrm{P}\ulcorner \underline{\forall x\, \exists y\, \mathrm{P}\ulcorner \underline{A(\dot{x}, \dot{y})} \urcorner} \urcorner.$$

So by ENEP for PEA + $\mathsf{F}^{\mathsf{i}}$, for every natural number $m$ there is a natural number $n$ such that PEA+$\mathsf{F}^{\mathsf{i}} \vdash \mathrm{P}\ulcorner \underline{A(\underline{m}, \underline{n})} \urcorner$, in which case $A(\underline{m}, \underline{n})$ is true in $\mathcal{M}$. So the Turing machine which enumerates $K$-proofs and halts when it sees a proof of the appropriate conclusion will do the job. ■

However, in light of what was said in the previous section, this appears to be in itself of little value. What really matters is whether ECT can be consistently included in the *inner* logic. But here one immediately runs into the fact that

**Proposition 11** PEA + $\mathsf{ECT}^{\mathsf{i}}$ *is inconsistent.*[11]

*Proof:* ECT implies full Reflexivity: just choose $A$ in such a way that it does not contain $x$ and $y$ free. ■

This leaves open the question whether the *restricted* version of ECT where the parameter $A(x, y)$ is allowed to range only over $\mathcal{L}_{\mathsf{PA}}$ can be consistently included in the inner logic. This question is not wholly devoid of significance, given the fact that this restricted version of ECT is arguably somewhat closer to the original Church's Thesis.

It is known that Gödel's translation from Heyting Arithmetic (HA) to Epistemic Arithmetic is sound and complete, i.e., faithful.[12] For this reason, Epistemic Arithmetic can be seen as an integration of classical and intuitionistic arithmetic. One can define an analogous mapping $g$ from HA to Predicate Epistemic Arithmetic. But now even the inductive argument for the *soundness* of this translation cannot be carried through. It appears that in order to carry out this inductive proof, C4 must be incorporated into the *inner logic* of PEA. But we have seen above (Theorem 8 on page 212) that PEA + $\mathsf{C4}^{\mathsf{i}}$ is inconsistent. It is not clear, therefore, how PEA-like systems can adequately simulate intuitionistic reasoning.

# 4 Informal provability and truth

The paradoxes about knowledge, necessity, truth and rational belief are produced by very similar diagonal arguments. In this sense, at least, these paradoxes are closely related. In this final section, I want to discuss the philosophical question that was left hitherto unaddressed: *why* is it that the notions of necessity, informal provability, and rational belief are subject to Liar-like paradoxes?

There is a view which holds that the relation between these paradoxes goes deeper.[13] The idea is that there is an underlying *conceptual connection* between the notions involved which explains *why* they are all paradoxical. This view is usually combined with the belief that at bottom there is only *one* paradoxical concept: truth. All the Liar-like paradoxes are just manifestations of the paradoxicality of the concept of truth.[14] But one wonders in which

[11] This observation is due to Volker Halbach.

[12] See Goodman 1984.

[13] This view appears to have numerous adherents. But it is rarely made explicit or defended in the literature.

[14] Perhaps this accounts for the fact that the semantic paradoxes have received much more attention than the Liar-like paradoxes.

way, exactly, this is so. For in the derivation of the Liar-like paradoxes the concept of truth does not occur.

One can attempt to argue that, contrary to first appearances, the concept of truth *does* play a role in the derivation of the Paradox of the Knower. In the previous section, we were moved to the tentative conclusion that most of the blame should go to the Reflexivity axiom. When we informally express the Reflexivity axiom in words, we often gloss it as "Whatever is provable is true", or "Provability entails truth". This means that we are appealing to something like the principle:

$$\forall x\,[\text{sentence}(x) \rightarrow (\mathrm{P}x \rightarrow \mathrm{T}x)], \quad \text{where T is a primitive truth predicate,}$$

to justify Reflexivity – let us call this principle PT. We only believe the Reflexivity axiom *because* we believe PT. But PT does contain the concept of truth. Since even the naïve Tarski-biconditionals are not all tenable, it should come as no surprise that the plausible-looking PT is after all untenable.

But this argument gives one a feeling similar to what Earman has expressed in an unrelated context (Earman 1995, 195):

> This is a pretty piece of ordinary language philosophizing. But like most of its ilk, it leaves one up in the air: even if one does share the linguistic intuitions, one can wonder how such intuitions can support such weighty philosophical morals.

The argument above appears to rest on a psycho-philosophical claim that is hard to conclusively establish or refute. But even if it proves to be fairly uncontroversial, it is indeed unclear what follows from it. For consider $\mathcal{L}_{\mathsf{PEA}}$ plus the new primitive truth predicate T. Take the following principles for P and T, against the background of PA (in the extended language):

1. $\forall x\,[\text{sentence}(x) \rightarrow (\mathrm{P}x \rightarrow \mathrm{T}x)]$;
2. $A \Rightarrow \mathrm{P}\ulcorner\underline{A}\urcorner$.

And assume with this, the *disquotational theory of truth* for $\mathcal{L}_{\mathsf{PEA}}$:

3. $\mathrm{T}\ulcorner\underline{A}\urcorner \leftrightarrow A \quad$ for all formulas $A \in \mathcal{L}_{\mathsf{PEA}}$.

Then the Paradox of the Knower still obtains. But *surely* truth here merely functions as a disquotational device. It here merely serves as a means for expressing infinite conjunctions. So it seems that there is a sense in which it is impossible to blame the Paradox of the Knower *solely* on the concept of truth. There is a sense in which the concept of informal provability is paradoxical in itself.

But perhaps this is going too fast. There may still be a way in which the notion of truth – or, more generally, semantical notions – can be said to be at the root of all the Liar-like paradoxes. In possible-worlds theory, truth is not seen as a property, but as a *relation* between sentences, possible worlds, places and moments in time. So instead of formalizing truth as a one-place predicate $\mathrm{T}(x)$, perhaps we ought really to formalize it as a *many-place* predicate $\mathrm{T}(x, w, t, i)$, which should be read as "$x$ is true in possible world $w$ at time $t$ and place $i$".

If this is on the right track, then the necessity predicate should be parsed as $\forall w \, \forall t \, \forall i \, \mathrm{T}(x, w, t, i)$. This would mean that Montague's Paradox involves the truth predicate after all. And since the notion of intuitive prov*ability* implicitly contains a modal component, it too would indirectly contain the concept of truth. Therefore the Paradox of the Knower would also involve the truth predicate.[15] Moreover, one would be led to suspect that temporal and spatial notions might also be subject to Liar-like paradoxes. And this is indeed the case.[16]

Thomason's paradox about rational belief remains. On the one hand, it is admittedly hard to see how rational belief contains a modal, a temporal or a spatial component. On the other hand, I submit that it is not completely clear that Thomason has presented us with a genuine paradox in the first place. The crucial axiom of his formalized theory for rational belief can be expressed as follows (where the predicate R expresses the notion of rational belief):

$$\mathrm{R}(\underline{\ulcorner \mathrm{R}(\underline{\ulcorner A \urcorner}) \rightarrow A \urcorner}) \quad \text{for every } A \text{, where R is allowed to occur in } A.$$

It is not clear to me that this axiom is intuitively plausible. We are sometimes in a situation where we have good reasons to believe a sentence $A$, but where $A$ is nevertheless false. We *know* that this is so. Therefore it seems that it would be more rational to suspend judgement on many sentences of the form $\mathrm{R}(\underline{\ulcorner A \urcorner}) \rightarrow A$ than to believe in all of them.

Of course this is just the *beginning* of a story of how the Liar-like paradoxes are related to the semantic paradoxes. And I am not sure how convincing it really is. Earman's worry is still ringing in my ears. After all, all we have is theorems about which kind of principles can be consistently combined with other principles. Isn't it just a contingent matter which informal notions appear in principles that are jointly for diagonalization reasons inconsistent? Isn't the quest for a unique source of the paradoxes futile?

[15] Note that the notion of *actually having* an informal proof is not, as far as we know, subject to Liar-like paradoxes.

[16] For a discussion of Liar-like paradoxes for tense logic, see Horsten and Leitgeb 2001.

## Appendix: Proof that PEA + $\mathsf{CF}^{\mathsf{i}}$ has an $\omega$-model

We start by considering partial models

$$\mathcal{M} = \langle \mathcal{M}(\mathrm{P}), \mathcal{M}(\mathrm{T})\rangle = \Big\langle \langle \mathcal{M}(\mathrm{P})^{+}, \mathcal{M}(\mathrm{P})^{-}\rangle, \langle \mathcal{M}(\mathrm{T})^{+}, \mathcal{M}(\mathrm{T})^{-}\rangle \Big\rangle$$

for $\mathcal{L}_{\mathsf{PEA}} \cup \{\mathrm{T}\}$, where T is an (auxiliary) truth predicate for $\mathcal{L}_{\mathsf{PEA}}$.

**Definition 12** *Let $\Phi$ denote the empty partial structure.*

**Definition 13** *A partial structure*

$$\mathcal{M} = \Big\langle \langle \mathcal{M}(\mathrm{P})^{+}, \mathcal{M}(\mathrm{P})^{-}\rangle, \langle \mathcal{M}(\mathrm{T})^{+}, \mathcal{M}(\mathrm{T})^{-}\rangle \Big\rangle$$

*is regular iff*

*(i)* $A, (A \rightarrow B) \in \mathcal{M}(\mathrm{P})^{+} \;\Rightarrow\; B \in \mathcal{M}(\mathrm{P})^{+}$;

*(ii)* $A \in \mathcal{M}(\mathrm{P})^{+} \;\Rightarrow\; \mathrm{P}\ulcorner \underline{A} \urcorner \in \mathcal{M}(\mathrm{P})^{+}$;

*(iii)* $\neg A \in \mathcal{M}(\mathrm{P})^{+} \;\Rightarrow\; \neg\mathrm{P}\ulcorner \underline{A} \urcorner \in \mathcal{M}(\mathrm{P})^{+}$.

**Definition 14** $\mathcal{M} \models_{\mathrm{sv}^*} A$ *if and only if for all regular $\mathcal{M}' \geq \mathcal{M}$ (where $\mathcal{M}' \geq \mathcal{M}$ is defined in the usual way), $\mathcal{M}' \models A$.*

**Proposition 15** sv* *is a monotone operation.*

*Proof:* Suppose $\mathcal{N} \geq \mathcal{M}$. If $\mathcal{M} \models_{\mathrm{sv}^*} A$ then for all regular $\mathcal{M}' \geq \mathcal{M}$, $\mathcal{M}' \models A$, and thus $\mathcal{M}' \models A$ for all regular $\mathcal{M}' \geq \mathcal{N}$. ∎

**Definition 16** *of a partial model $\mathcal{I}_\omega$, based on the natural number structure, in stages $n \leq \omega$.*

*(i)* $\mathcal{I}(\mathrm{P})^{+}_{0} = \mathsf{BPEA} + \mathsf{CF}$;

*(ii)* $\mathcal{I}(\mathrm{P})^{-}_{n} = \mathcal{I}(\mathrm{T})^{-}_{n}$ *for all $n < \omega$;*

*(iii)* $\mathcal{I}(\mathrm{T})^{+}_{0} = \{A \mid \Phi \models_{\mathrm{sv}^*} A\}$;

*(iv)* $\mathcal{I}(\mathrm{T})^{-}_{0} = \{A \mid \Phi \models_{\mathrm{sv}^*} \neg A\}$;

*(v)* $\mathcal{I}(\mathrm{T})^{+}_{n+1} = \{A \mid \mathcal{I}_n \models_{\mathrm{sv}^*} A\}$, *where $\mathcal{I}_n$ is the partial structure defined by stage $n$;*

*(vi)* $\mathcal{I}(\mathrm{T})^{-}_{n+1} = \{A \mid \mathcal{I}_n \models_{\mathrm{sv}^*} \neg A\}$;

*(vii)* $\mathcal{I}(\mathrm{P})^+_{n+1}$ *is the closure under Modus Ponens of*

$$\mathcal{I}(\mathrm{P})^+_n \cup \{\, \mathrm{P}\ulcorner\underline{A}\urcorner \mid A \in \mathcal{I}(\mathrm{P})^+_n \,\} \cup \{\, \neg\mathrm{P}\ulcorner\underline{A}\urcorner \mid \neg A \in \mathcal{I}(\mathrm{P})^+_n \,\};$$

*(viii)* $\mathcal{I}(\mathrm{P})^\pm_\omega = \bigcup_{n<\omega} \mathcal{I}(\mathrm{P})^\pm_n$;

*(ix)* $\mathcal{I}(\mathrm{T})^\pm_\omega = \bigcup_{n<\omega} \mathcal{I}(\mathrm{T})^\pm_n$.

**Proposition 17** *For all n,* $\mathcal{I}(\mathrm{T})^+_n \subseteq \mathcal{I}(\mathrm{T})^+_{n+1}$.

*Proof:* Straightforward induction on $n$, using the monotonicity of sv*. ■

**Proposition 18** $\mathcal{I}(\mathrm{P})^+_0 \subseteq \mathcal{I}(\mathrm{T})^+_0$.

*Proof:* Induction on the length of proofs in the *basis* of PEA plus CF.

*Basis of the induction:*

(i) PA $\subseteq \mathcal{I}(\mathrm{T})^+_0$ because $\mathcal{I}(\mathrm{T})^+_0$ is based on the standard natural number structure.

(ii) For the other axioms we use the regularity conditions of sv*.

*Induction step:* Trivial.

■

**Proposition 19** *For all n,* $\mathcal{I}(\mathrm{P})^+_n \subseteq \mathcal{I}(\mathrm{T})^+_n$.

*Proof:* Induction on stages $n$.

*Basis of the induction:* This is taken care of by the previous proposition.

*Induction step:* By induction on "length of proofs" in $\mathcal{I}(\mathrm{P})^+_{n+1}$.

(a) Basis:

(i) $A \in \mathcal{I}(\mathrm{P})^+_n \overset{\text{I.H.}}{\Rightarrow} A \in \mathcal{I}(\mathrm{T})^+_n \overset{\text{Prop. 17}}{\Rightarrow} A \in \mathcal{I}(\mathrm{T})^+_{n+1}$.

(ii) $A \in \mathcal{I}(\mathrm{P})^+_n \Rightarrow \mathcal{I}_n \models_{\text{sv*}} \mathrm{P}\ulcorner\underline{A}\urcorner \Rightarrow \mathrm{P}\ulcorner\underline{A}\urcorner \in \mathcal{I}(\mathrm{T})^+_{n+1}$.

(iii) $\neg A \in \mathcal{I}(\mathrm{P})^+_n \overset{\text{I.H.}}{\Rightarrow} \neg A \in \mathcal{I}(\mathrm{T})^+_n \Leftrightarrow A \in \mathcal{I}(\mathrm{T})^-_n \Leftrightarrow A \in \mathcal{I}(\mathrm{P})^-_n \Rightarrow \mathcal{I}_n \models_{\text{sv*}} \neg\mathrm{P}\ulcorner\underline{A}\urcorner \Rightarrow \neg\mathrm{P}\ulcorner\underline{A}\urcorner \in \mathcal{I}(\mathrm{T})^+_{n+1}$.

(b) Induction step: $\mathcal{I}(\mathrm{P})^+_n$ is closed under Modus Ponens, for all $n$.

■

**Proposition 20** *For all n,*

$$\mathcal{I}(\mathrm{P})^+_n \cap \mathcal{I}(\mathrm{P})^-_n = \mathcal{I}(\mathrm{P})^+_n \cap \mathcal{I}(\mathrm{T})^-_n = \mathcal{I}(\mathrm{T})^+_n \cap \mathcal{I}(\mathrm{T})^-_n = \emptyset.$$

*Proof:* Induction on stages $n$.

*Basis of the induction:* We know that $\mathcal{I}(\mathrm{P})_0^+ \subseteq \mathcal{I}(\mathrm{T})_0^+$ (Proposition 18) and $\mathcal{I}(\mathrm{P})_0^- = \mathcal{I}(\mathrm{T})_0^-$. So it suffices to show that $\mathcal{I}(\mathrm{T})_0^+ \cap \mathcal{I}(\mathrm{T})_0^- = \emptyset$, but that is obvious.

*Induction step:* We are given that

$$\mathcal{I}(\mathrm{P})_n^+ \cap \mathcal{I}(\mathrm{P})_n^- = \mathcal{I}(\mathrm{P})_n^+ \cap \mathcal{I}(\mathrm{T})_n^- = \mathcal{I}(\mathrm{T})_n^+ \cap \mathcal{I}(\mathrm{T})_n^- = \emptyset,$$

and want to show that

$$\mathcal{I}(\mathrm{P})_{n+1}^+ \cap \mathcal{I}(\mathrm{P})_{n+1}^- = \mathcal{I}(\mathrm{P})_{n+1}^+ \cap \mathcal{I}(\mathrm{T})_{n+1}^- = \mathcal{I}(\mathrm{T})_{n+1}^+ \cap \mathcal{I}(\mathrm{T})_{n+1}^- = \emptyset.$$

We know by Proposition 19 that $\mathcal{I}(\mathrm{P})_{n+1}^+ \subseteq \mathcal{I}(\mathrm{T})_{n+1}^+$, and we know that $\mathcal{I}(\mathrm{P})_{n+1}^- = \mathcal{I}(\mathrm{T})_{n+1}^-$. But by the induction hypothesis and Clauses (v), (vi) of Definition 16, $\mathcal{I}(\mathrm{T})_{n+1}^+ \cap \mathcal{I}(\mathrm{T})_{n+1}^-$ must be $\emptyset$, so we are done.

■

From this last proposition it follows that we can close off $\mathcal{I}_\omega$ to obtain a Tarskian structure $\mathcal{I}_\omega^c$. Note that by the construction of the model $\mathcal{I}_\omega$ (see its successor clause for the extension of P), $\mathcal{I}_\omega^c$ is *regular.* We now claim that

**Theorem 21** $\mathcal{I}_\omega^c \models \mathsf{PEA} + \mathsf{CF}^{\mathsf{i}}$.

*Proof:* Induction on the length of proofs in $\mathsf{PEA} + \mathsf{CF}^{\mathsf{i}}$.

*Basis of the induction:*

(i) $\mathcal{I}_\omega^c \models \mathsf{PA}$. This follows from the fact that the underlying arithmetical structure is the standard model.

(ii) $\mathcal{I}_\omega^c \models \mathrm{Bew}_{\mathsf{BPEA+CF}}(\ulcorner \underline{A} \urcorner) \rightarrow \mathrm{P}\ulcorner \underline{A} \urcorner$. Suppose $\mathcal{I}_\omega^c \models \mathrm{Bew}_{\mathsf{BPEA+CF}}(\ulcorner \underline{A} \urcorner)$. Then $\mathsf{BPEA} + \mathsf{CF} \vdash A$. Since $\mathcal{I}(\mathrm{P})_0^+ = \mathsf{BPEA} + \mathsf{CF}$, we therefore have $A \in \mathcal{I}(\mathrm{P})_0^+$. Therefore $A \in \mathcal{I}(\mathrm{P})_\omega^+$. So $\mathcal{I}_\omega \models_{\mathsf{sv}^*} \mathrm{P}\ulcorner \underline{A} \urcorner$, whereby $\mathcal{I}_\omega^c \models \mathrm{P}\ulcorner \underline{A} \urcorner$.

(iii) $\mathcal{I}_\omega^c \models \mathrm{P}\ulcorner \underline{A} \urcorner \rightarrow A$. Suppose $\mathcal{I}_\omega^c \models \mathrm{P}\ulcorner \underline{A} \urcorner$. Then $A \in \mathcal{I}(\mathrm{P})_n^+$ for some $n$. Then $A \in \mathcal{I}(\mathrm{T})_n^+$. Therefore $\mathcal{I}_\omega \models_{\mathsf{sv}^*} A$. So, by the regularity of $\mathcal{I}_\omega^c$, it follows that $\mathcal{I}_\omega^c \models A$.

*Induction step:* $\mathcal{I}_\omega^c$ is closed under Modus Ponens.

■

# Bibliography


Aczel, Peter. 1977. "The strength of Martin-Löf's intuitionistic type theory with one universe." In *Proceedings of the Symposiums in Mathematical Logic in Oulu 1974 and Helsinki 1975*, edited by S. Mietissen and J. Väänänen, 1–32. Report no. 2.

Azzouni, Jody. 1999. "Comments on Shapiro." *Journal of Philosophy* 96 (10): 541–44 (October).

Barker, John. 1998. "The Inconsistency Theory of Truth." Ph.D. diss., Princeton University, Princeton.

Barwise, Jon and John Etchemendy. 1987. *The Liar: An Essay on Truth and Circularity*. New York and Oxford: Oxford University Press.

Belnap, Nuel D. 1962. "Tonk, Plonk and Plink." *Analysis* 22 (6): 130–34 (June). Reprinted in *Philosophical Logic*, edited by P[eter] F[rederick] Strawson, 132–37; London: Oxford University Press, 1967.

———. 1982. "Gupta's Rule of Revision Theory of Truth." *Journal of Philosophical Logic* 11:103–16.

Berka, Karel and Lothar Kreiser. 1986. *Logik-Texte: Kommentierte Auswahl zur Geschichte der modernen Logik*. Fourth, revised edition. Berlin: Akademie-Verlag. First edition 1971, third, enlarged edition 1983.

Boolos, George. 1984. "To Be is to Be a Value of a Variable (or to Be Some Values of Some Variables)." *Journal of Philosophy* 81 (8): 430–49 (August). Reprinted in Boolos 1998, 54–72.

———. 1985. "Nominalist Platonism." *Philosophical Review* 94 (3): 327–44 (July). Reprinted in Boolos 1998, 73–87.

———. 1998. *Logic, Logic, and Logic*. Cambridge (Mass.) and London: Harvard University Press. Edited by Richard Jeffrey, with introductions and afterword by John P. Burgess.

Boolos, George S. and Richard C. Jeffrey. 1989. *Computability and Logic*. Third edition. Cambridge, New York, Port Chester, Melbourne and Sydney: Cambridge University Press.

Braithwaite, Richard Bevan. 1953. *Scientific Explanation: A Study of the Function of Theory, Probability and Law in Science.* Cambridge: Cambridge University Press.

Brandom, Robert. 1987. "Pragmatism, Phenomenalism, and Truth Talk." *Midwest Studies in Philosophy* 12:75–93.

Brendel, Elke. 1992. *Die Wahrheit über den Lügner: Eine philosophisch-logische Analyse der Antinomie des Lügners.* Berlin and New York: De Gruyter.

Buchholz, Wilfried. 1997. "An intuitionistic fixed point theory." *Archive for Mathematical Logic* 37:21–27.

Burgess, John P. 1981. "Relevance: A Fallacy?" *Notre Dame Journal of Formal Logic* 22 (2): 97–104 (April).

———. 1986. "The Truth Is Never Simple." *Journal of Symbolic Logic* 51 (3): 663–81 (September).

———. 1988. "Addendum to 'The Truth Is Never Simple'." *Journal of Symbolic Logic* 53 (2): 390–92 (June).

Cantini, Andrea. "Paradoxes." Forthcoming in *A History of Mathematical Logic*, edited by Dirk van Dalen et al., Elsevier.

———. 1989. "Notes on Formal Theories of Truth." *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 35:97–130.

———. 1990. "A Theory of Formal Truth Arithmetically Equivalent to $\mathsf{ID}_1$." *Journal of Symbolic Logic* 55 (1): 244–59 (March).

———. 1996. *Logical Frameworks for Truth and Abstraction: An Axiomatic Study.* Studies in Logic and the Foundations of Mathematics no. 135. Amsterdam, Lausanne, New York, Oxford, Shannon and Tokyo: Elsevier.

Casalegno, Paolo. "Some Remarks on Deflationism." Unpublished manuscript, University of Milan.

Casari, Ettore. 1989. "Comparative Logics and Abelian *l*-Groups." In *Logic Colloquium '88: Proceedings of the Colloquium held in Padova, Italy, August 22–31, 1988*, edited by R. Ferro, C. Bonotto, S. Valentini, and A. Zanardo, 161–90. Amsterdam, New York, Oxford and Tokyo: North-Holland.

Chihara, Charles. 1979. "The Semantic Paradoxes: A Diagnostic Investigation." *Philosophical Review* 88 (4): 590–618 (October).

Corcoran, John. 1980. "Categoricity." *History and Philosophy of Logic* 1:187–207.

Corcoran, John, William Frank, and Michael Maloney. 1974. "String Theory." *Journal of Symbolic Logic* 39 (4): 625–37 (December).

David, Marian. 1994. *Correspondence and Disquotation: An Essay on the Nature of Truth.* New York and Oxford: Oxford University Press.

Davidson, Donald. 1967. "Truth and Meaning." *Synthese* 17:304–23. Reprinted in Davidson 1984, 17–36.

———. 1977. "Reality Without Reference." *Dialectica* 31 (3–4): 247–58. Reprinted in Davidson 1984, 215–25; page references are given for the reprint.

———. 1984. *Inquiries into Truth and Interpretation.* Oxford: Clarendon Press. Reprinted with corrections, 1990.

———. 1996. "The Folly of Trying to Define Truth." *Journal of Philosophy* 93 (6): 263–78 (June).

Diogenes Laertius. 1972. *Lives of Eminent Philosophers.* Revised edition. Loeb Classical Library. Translated by R[obert] D[rew] Hicks. London and Cambridge (Mass.): William Heinemann and Harvard University Press. Revised ed. Herbert S. Long.

Duhem, Pierre. 1954. *The Aim and Structure of Physical Theory.* Translated from the French by Philip P. Wiener. Princeton: Princeton University Press. First published as *La Théorie physique: Son objet – sa structure* (Second edition) in Paris: Marcel Rivière, 1914.

Dummett, Michael. 1958–59. "Truth." *Proceedings of the Aristotelian Society* 59:141–62.

Dunn, J. Michael and Nuel D. Belnap, Jr. 1968. "The Substitution Interpretation of the Quantifiers." *Noûs* 2:177–85.

Earman, John. 1995. *Bangs, Crunches, Whimpers, and Shrieks: Singularities and Acausalities in Relativistic Spacetimes.* New York and Oxford: Oxford University Press.

Etchemendy, John W. 1999. *The Concept of Logical Consequence.* Second edition. Stanford: CSLI. First edition Cambridge (Mass.): Harvard University Press, 1990.

Feferman, Solomon. 1964. "Systems of Predicative Analysis." *Journal of Symbolic Logic* 29 (1): 1–30 (March).

———. 1979. "Constructive theories of functions and classes." In *Logic Colloquium '78: Proceedings of the Colloquium held in Mons, August 1978,* edited by Maurice Boffa, Dirk van Dalen, and Kenneth McAloon, 159–224. Amsterdam, New York and Oxford: North-Holland.

———. 1984. "Towards Useful Type-free Theories. I." *Journal of Symbolic Logic* 49 (1): 75–111 (March).

———. 1991. "Reflecting on Incompleteness." *Journal of Symbolic Logic* 56 (1): 1–49 (March).

Field, Hartry. 1972. "Tarski's Theory of Truth." *Journal of Philosophy* 69 (13): 347–75 (July).

———. 1986. "The Deflationary Conception of Truth." In *Fact, Science and Morality: Essays on A. J. Ayer's* Language, Truth and Logic, edited by Graham Macdonald and Crispin Wright, 55–117. Oxford and New York: Basil Blackwell.

———. 1992. "Critical Notice: Paul Horwich's *Truth*." *Philosophy of Science* 59:321–30.

———. 1994a. "Deflationist Views of Meaning and Content." *Mind* 103 (411): 247–85 (July).

———. 1994b. "Disquotational Truth and Factually Defective Discourse." *Philosophical Review* 103 (3): 405–52 (July).

———. 1998. "Which undecidable sentences have determinate truth values." In *Truth in Mathematics*, edited by H. G. Dales and G. Oliveri, 291–301. New York: Oxford University Press.

———. 1999. "Deflating the Conservativeness Argument." *Journal of Philosophy* 96 (10): 533–40 (October).

Fitch, Frederic Brenton. 1936. "A System of Formal Logic Without an Analogue to the Curry *W* Operator." *Journal of Symbolic Logic* 1 (3): 92–100 (September).

———. 1963. "A Logical Analysis of Some Value Concepts." *Journal of Symbolic Logic* 28 (2): 135–42 (June).

Friedman, Harvey and Michael Sheard. 1987. "An Axiomatic Approach to Self-Referential Truth." *Annals of Pure and Applied Logic* 33:1–21.

———. 1988. "The Disjunction and Existence Properties for Axiomatic Systems of Truth." *Annals of Pure and Applied Logic* 40:1–10.

Germano, Giorgio. 1970. "Metamathematische Begriffe in Standardtheorien." *Archiv für Mathematische Logik* 13:22–38.

Girard, Jean-Yves. 1987. "Linear Logic." *Theoretical Computer Science* 50:1–101.

Goldblatt, Robert. 1992. *Logics of Time and Computation*. Second edition, revised and expanded. CSLI Lecture Notes no. 7. Stanford: CSLI.

Goodman, Nicolas D. 1984. "Epistemic Arithmetic is a Conservative Extension of Intuitionistic Arithmetic." *Journal of Symbolic Logic* 49 (1): 192–203 (March).

Grišin, V. N. 1973. "A Non-standard Logic, and its Application to Set Theory." In *Studies in Formalized Languages and Non-classical Logic*, 135–71 (in Russian). Moscow.

Grover, Dorothy L., Joseph L. Camp, Jr., and Nuel D. Belnap, Jr. 1975. "A Prosentential Theory of Truth." *Philosophical Studies* 27:73–125.

Gupta, Anil. 1982. "Truth and Paradox." *Journal of Philosophical Logic* 11:1–60.

———. 1993a. "A Critique of Deflationism." *Philosophical Topics* 21 (2): 57–81 (Fall).

———. 1993b. "Minimalism." In *Language and Logic*, edited by James E. Tomberlin, Volume 7 of *Philosophical Perspectives*, 359–69. Atascadero: Ridgeview Publishing.

Gupta, Anil and Nuel Belnap. 1993. *The Revision Theory of Truth*. Cambridge (Mass.) and London: MIT Press.

Hájek, Petr and Pavel Pudlák. 1993. *Metamathematics of First-Order Arithmetic*. Perspectives in Mathematical Logic. Berlin, Heidelberg, New York etc.: Springer.

Halbach, Volker. 1994. "A System of Complete and Consistent Truth." *Notre Dame Journal of Formal Logic* 35 (3): 311–27 (Summer).

———. 1996. *Axiomatische Wahrheitstheorien*. Berlin: Akademie Verlag.

———. 1999a. "Conservative Theories of Classical Truth." *Studia Logica* 62 (3): 353–70.

———. 1999b. "Disquotationalism and Infinite Conjunctions." *Mind*, n. s. 108 (429): 1–22 (January).

———. 2000a. "Disquotationalism Fortified." In *Circularity, Definitions, and Truth*, edited by André Chapuis and Anil Gupta, 155–76. New Delhi: Indian Council of Philosophical Research, Munshiram Manoharlal Publishers.

———. 2000b. "Truth and Reduction." *Erkenntnis* 53:97–126.

———. 2001a. "Disquotational Truth and Analyticity." *Journal of Symbolic Logic* 66 (4): 1959–73 (December).

———. 2001b. "How Innocent Is Deflationism?" *Synthese* 126 (1–2): 167–94 (January/February).

———. 2001c. "Semantics and Deflationism." Unpublished Habilitationsschrift.

Harris, J[ohn] H. 1982. "What's So Logical about the 'Logical' Axioms?" *Studia Logica* 41 (2–3): 159–71.

Hellman, Geoffrey. 1985. Review of Martin and Woodruff 1975, Kripke 1975, Gupta 1982 and Herzberger 1982. *Journal of Symbolic Logic* 50 (4): 1068–71 (December).

Henkin, Leon. 1950. "Completeness in the Theory of Types." *Journal of Symbolic Logic* 15 (2): 81–91 (June).

Herzberger, Hans G. 1982. "Notes on Naive Semantics." *Journal of Philosophical Logic* 11:61–102.

Hodges, Wilfrid. 1986. "Truth in a Structure." *Proceedings of the Aristotelean Society*, n. s. 86:135–51.

Horsten, Leon. 1997. "Provability in Principle and Controversial Constructivistic Principles." *Journal of Philosophical Logic* 26:635–60.

Horsten, Leon and Hannes Leitgeb. 2001. "No Future." *Journal of Philosophical Logic* 30 (3): 259–65 (June).

Horwich, Paul. 1990. *Truth*. Oxford: Basil Blackwell.

———. 1994. *Theories of Truth*. Dartmouth: Aldershot.

———. 1997a. "The Composition of Meanings." *Philosophical Review* 106 (4): 503–32 (October).

———. 1997b. "Implicit Definition, Analytic Truth, and Apriori Knowledge." *Noûs* 31 (4): 423–40.

———. 1998a. *Meaning*. Oxford: Clarendon Press.

———. 1998b. *Truth*. Second edition. Oxford: Oxford University Press.

———. 2000. "Stipulation, Meaning, and Apriority." In *New Essays on the A Priori*, edited by Paul Boghossian and Christopher Peacocke, 150–69. Oxford: Clarendon Press.

———. 2001a. "A Defense of Minimalism." *Synthese* 126:149–65.

———. 2001b. "Deflating Compositionality." *Ratio*, n. s. 14 (4): 369–85 (December).

———. 2002. "Norms of Truth and Meaning." In *What is Truth?*, edited by Richard Schantz, Volume 1 of *Current Issues in Theoretical Philosophy*, 133–45. Berlin and New York: De Gruyter.

Jäger, Gerhard, Reinhard Kahle, Anton Setzer, and Thomas Strahm. 1999. "The Proof-Theoretic Analysis of Transfinitely Iterated Fixed Point Theories." *Journal of Symbolic Logic* 64 (1): 53–67 (March).

James, William. 1978. *Pragmatism: A New Name for Some Old Ways of Thinking*. Cambridge (Mass.) and London: Harvard University Press. One-volume issue with *The Meaning of Truth: A Sequel to* Pragmatism.

Kahle, Reinhard. 1997. "Applikative Theorien und Frege-Strukturen." Ph.D. diss., IAM, Bern.

Kaplan, David and Richard Montague. 1960. "A Paradox Regained." *Notre Dame Journal of Formal Logic* 1:79–90. Reprinted in Montague 1974, 271–85.

Ketland, Jeffrey. 1999. "Deflationism and Tarski's Paradise." *Mind* 108 (429): 69–94 (January).

Ketonen, Jussi and Richard Weyhrauch. 1984. "A decidable fragment of predicate calculus." *Theoretical Computer Science* 32:297–307.

Kotlarski, Henryk and Zygmunt Ratajczyk. 1990a. "Inductive Full Satisfaction Classes." *Annals of Pure and Applied Logic* 47:199–223.

———. 1990b. "More on Induction in the Language with a Satisfaction Class." *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 36:441–54.

Kotlarski, H[enryk], S[tanislav] Krajewski, and A[listair] H. Lachlan. 1981. "Construction of Satisfaction Classes for Nonstandard Models." *Canadian Mathematical Bulletin* 24 (3): 283–93 (September).

Krajewski, S[tanislav]. 1976. "Non-standard Satisfaction Classes." In *Set Theory and Hierarchy Theory: A Memorial Tribute to Andrzej Mostowski, Bierutowice, Poland 1975*, edited by W[iktor] Marek, M. Srebrny, and A. Zarach, Lecture Notes in Mathematics no. 537, 121–44. Berlin, Heidelberg and New York: Springer.

Kraut, Robert. 1993. "Robust Deflationism." *Philosophical Review* 102 (2): 247–63 (April).

Kreisel, Georg. 1987. "Gödel's Excursions into Intuitionistic Logic." In *Gödel Remembered, Salzburg 10–12 July 1983*, edited by Paul Weingartner and Leopold Schmetterer, 77–120. Naples: Bibliopolis.

———. 1992. "On the idea(l) of logical closure." *Annals of Pure and Applied Logic* 56:19–41.

Kripke, Saul. 1975. "Outline of a Theory of Truth." *Journal of Philosophy* 72 (19): 690–716 (November).

———. 1976. "Is There a Problem about Substitutional Quantification?" In *Truth and Meaning: Essays in Semantics*, edited by Gareth Evans and John McDowell, 325–419. Oxford: Clarendon Press.

———. 1982. *Wittgenstein on Rules and Private Language: An Elementary Exposition.* Cambridge (Mass.): Harvard University Press.

Lachlan, A[listair] H. 1981. "Full Satisfaction Classes and Recursive Saturation." *Canadian Mathematical Bulletin* 24 (3): 295–97 (September).

Leitgeb, Hannes. 1999a. "Truth and the Liar in De Morgan–Valued Models." *Notre Dame Journal of Formal Logic* 40 (4): 496–514.

———. 1999b. "Truth as Translation." Forschungsbericht der Forschergruppe *Logik in der Philosophie* 44, Universität Konstanz and Eberhard-Karls-Universität Tübingen.

Lewy, C[asimir]. 1947. "Truth and Significance." *Analysis* 8 (2): 24–27 (December).

Martin, Robert L., ed. 1984. *Recent Essays on Truth and the Liar Paradox.* Oxford and New York: Clarendon Press and Oxford University Press.

Martin, Robert L. and Peter W. Woodruff. 1975. "On Representing 'True-in-*L*' in *L*." *Philosophia* 5 (3): 213–17 (July).

McGee, Vann. 1985. "How Truthlike Can a Predicate Be? A Negative Result." *Journal of Philosophical Logic* 14:399–410.

———. 1991. *Truth, Vagueness, and Paradox: An Essay on the Logic of Truth.* Indianapolis and Cambridge: Hackett Publishing.

———. 1993. "A Semantic Conception of Truth?" *Philosophical Topics* 21 (2): 83–111 (Fall).

———. 1997. "How We Learn Mathematical Language." *Philosophical Review* 106 (1): 35–68 (January).

———. 2000. "'Everything'." In *Between Logic and Intuition: Essays in Honor of Charles Parsons*, edited by Gila Sher and Richard Tieszen, 54–78. Cambridge: Cambridge University Press.

———. 2001. "Truth by Default." *Philosophia Mathematica* (3) 9 (1): 5–20 (February).

McGee, Vann and Brian P. McLaughlin. 2000. "The Lessons of the Many." *Philosophical Topics* 28 (1): 129–51 (Spring).

McGrath, Matthew. 1997. "Weak Deflationism." *Mind* 106 (421): 69–98 (January).

Montague, Richard. 1963. "Syntactical Treatments of Modality, with Corollaries on Reflexion Principles and Finite Axiomatizability." *Acta Philosophica Fennica* 16:153–67. Reprinted in Montague 1974, 286–302.

———. 1974. *Formal Philosophy: Selected Papers of Richard Montague.* New Haven and London: Yale University Press. Edited and with an introduction by Richmond H. Thomason.

Moschovakis, Yiannis N. 1974. *Elementary Induction on Abstract Structures.* Studies in Logic and the Foundations of Mathematics no. 77. Amsterdam, London and New York: North-Holland and Elsevier.

Mostowski, Andrzej. 1950. "Some Impredicative Definitions in the Axiomatic Set-Theory." *Fundamenta Mathematicae* 37:111–24.

Myhill, John. 1950. "A system which can define its own truth." *Fundamenta Mathematicae* 37:190–92.

———. 1960. "Some Remarks on the Notion of Proof." *Journal of Philosophy* 57 (14): 461–71 (July).

Niebergall, Karl-Georg. 1991. "Simultane objektsprachliche Axiomatisierung von Notwendigkeits- und Beweisbarkeitsprädikaten." Master's thesis, Ludwigs-Maximilians-Universität München.

Ono, Hiroakira and Yuichi Komori. 1985. "Logics without the Contraction Rule." *Journal of Symbolic Logic* 50 (1): 169–201 (March).

Parsons, Charles. 1974. "The Liar Paradox." *Journal of Philosophical Logic* 3:381–412.

———. 1990. "The Uniqueness of the Natural Numbers." *Iyyun* 39:13–44.

Prawitz, Dag. 1994. *Natural Deduction: A Proof-Theoretical Study.* Acta Universitatis Stockholmiensis/Stockholm Studies in Philosophy no. 3. Stockholm, Göteborg and Uppsala: Almqvist & Wiksell.

Putnam, Hilary. 1985. "A Comparison of Something with Something Else." *New Literary History* 17 (1): 61–79 (Autumn). Reprinted in Putnam 1994, 330–50.

———. 1994. *Words and Life.* Cambridge (Mass.) and London: Harvard University Press. Edited by James Conant.

Quine, Willard Van Orman. 1960. *Word and Object.* Cambridge (Mass.): MIT Press.

———. 1968. "Ontological Relativity." *Journal of Philosophy* 65 (7): 185–212 (April). Reprinted in Quine 1969, 26–68.

———. 1969. *Ontological Relativity and Other Essays.* The John Dewey Essays in Philosophy no. 1. New York and London: Columbia University Press.

———. 1977. "Intensions Revisited." In *Contemporary Perspectives in the Philosophy of Language*, edited by Peter A. French, Theodore E. Uehling, jr., and Howard K. Wettstein, 5–11. Minneapolis: University of Minnesota Press.

———. 1986. *Philosophy of Logic.* Second edition. Foundations of Philosophy. Cambridge (Mass.) and London: Harvard University Press. First edition Englewood Cliffs: Prentice-Hall, 1970.

Ramsey, F[rank] P[lumpton]. 1927. "Facts and Propositions." *Proceedings of the Aristotelean Society Supplement* 7 (July): 153–70. Reprinted in Ramsey 1931b, 138–55; Ramsey 1978, 40–57; Ramsey 1990, 34–51; page references are to Ramsey 1990.

———. 1931a. "Causal Qualities." In Ramsey 1931b, 260–62. Reprinted in Ramsey 1990, 137–39.

———. 1931b. *The Foundations of Mathematics and other Logical Essays.* London: Routledge & Kegan Paul. Edited by R[ichard] B[evan] Braithwaite; fourth printing 1965.

———. 1931c. "Theories." In Ramsey 1931b, 212–36. Reprinted in Ramsey 1978, 101–25; Ramsey 1990, 112–36.

———. 1931d. "Truth and Probability." In Ramsey 1931b, 156–98. Reprinted in Ramsey 1978, 58–100; Ramsey 1990, 52–94; page references are to Ramsey 1990.

———. 1978. *Foundations: Essays in Philosophy, Logic, Mathematics and Economy.* London and Henley: Routledge & Kegan Paul. Edited by D[avid] H. Mellor.

———. 1990. *Philosophical Papers.* Cambridge, New York, Port Chester, Melbourne and Sydney: Cambridge University Press. Edited by D[avid] H. Mellor.

Rasmussen, Knud. 1931. *The Netsilik Eskimos: Social Life and Spiritual Culture. Report of the Fifth Thule Expedition.* Volume 8. Copenhagen: Gyldendalske Boghadel.

Reinhardt, William N. 1980. "Necessity Predicates and Operators." *Journal of Philosophical Logic* 9:437–50.

———. 1986. "Some Remarks on Extending and Interpreting Theories with a Partial Predicate for Truth." *Journal of Philosophical Logic* 15:219–51.

Resnik, Michael D. 1997. *Mathematics as a Science of Patterns.* Oxford: Clarendon Press.

Richard, Mark. 1997. "Deflating Truth." In *Truth,* edited by Enrique Villanueva, Volume 8 of *Philosophical Issues,* 57–78. Atascadero: Ridgeview Publishing.

Schütte, Kurt. 1977. *Proof Theory.* Grundlehren der mathematischen Wissenschaften no. 225. Translated from the German by J. N. Crossley. Berlin, Heidelberg and New York: Springer. First published as *Beweistheorie* in 1960. Grundlehren der mathematischen Wissenschaften 103, rev. ed.

Schwichtenberg, Helmut. 1977. "Proof Theory: Some Applications of Cut-Elimination." In *Handbook of Mathematical Logic,* edited by Jon Barwise, Studies in Logic and the Foundations of Mathematics no. 90, 867–95. Amsterdam, London, New York and Tokyo: North-Holland.

Shapiro, Stewart. 1985a. "Epistemic and Intuitionistic Arithmetic." In Shapiro 1985b, 11–46.

———, ed. 1985b. *Intensional Mathematics.* Studies in Logic and the Foundations of Mathematics no. 113. Amsterdam, New York and Oxford: North-Holland.

———. 1985c. "Second-Order Languages and Mathematical Practice." *Journal of Symbolic Logic* 50 (3): 714–42 (September).

———. 1991. *Foundations without Foundationalism: A Case for Second-order Logic.* Oxford Logic Guides no. 17. Oxford: Clarendon Press.

———. 1998. "Proof and Truth: Through Thick and Thin." *Journal of Philosophy* 95 (10): 493–521 (October).

———. 1999. "The Guru, the Logician, and the Deflationist." Submitted.

———. 2001. "Systems between First-order and Second-order Logics." In *Handbook of Philosophical Logic,* edited by Dov M. Gabbay and Franz Guenthner, Volume 1, Second edition, 131–87. The Netherlands: Kluwer Academic Publishers.

Sheard, Michael. 1994. "A Guide to Truth Predicates in the Modern Era." *Journal of Symbolic Logic* 59 (3): 1032–54 (September).

Smith, Jan. 1984. "An Interpretation of Martin-Löf's Type Theory in a Type-free Theory of Propositions." *Journal of Symbolic Logic* 49 (3): 730–53 (September).

Smoryński, Craig. 1985. *Self-Reference and Modal Logic.* Universitext. New York, Berlin, Heidelberg and Tokyo: Springer.

Soames, Scott. 1997. "The Truth about Deflationism." In *Truth*, edited by Enrique Villanueva, Volume 8 of *Philosophical Issues*, 1–44. Atascadero: Ridgeview Publishing.

———. 1999. *Understanding Truth.* New York and Oxford: Oxford University Press.

Sosa, Ernest. 1993. "The Truth of Modest Realism." *Philosophical Issues* 3:177–95.

Strawson, P[eter] F[rederick]. 1949. "Truth." *Analysis* 9 (6): 83–97 (June).

Takeuti, Gaisi. 1987. *Proof Theory.* Second edition. Volume 81 of *Studies in Logic and the Foundations of Mathematics.* Amsterdam, New York, Oxford and Tokyo: North-Holland.

Tarski, Alfred. 1935. "Der Wahrheitsbegriff in den formalisierten Sprachen." *Studia Philosophica* 1:261–405. Reprinted as "The Concept of Truth in Formalized Languages" in Tarski 1956, 152–278; page references are given for the translation.

———. 1936a. "O pojeciu wynikania logicznego." *Przegląd Filozoficzny* (= Révue philosophique) 39:58–68.

———. 1936b. "Über den Begriff der logischen Folgerung." *Actes du congrès international de philosophie scientifique* 7:1–11. Translation of Tarski 1936a. Reprinted (abbreviated) in Berka and Kreiser 1986, 404–13; also reprinted as "On the Concept of Logical Consequence" in Tarski 1983, 409–20; page references are given for the English translation.

———. 1944. "The Semantic Conception of Truth and the Foundations of Semantics." *Philosophy and Phenomenological Research* 4 (3): 341–76 (March).

———. 1956. *Logic, Semantics, Metamathematics: Papers from 1923 to 1938.* Translated by J[oseph] H. Woodger. Oxford: Clarendon Press.

———. 1983. *Logic, Semantics, Metamathematics: Papers from 1923 to 1938.* Second edition. Translated by J[oseph] H. Woodger. Indianapolis: Hackett Publishing. Edited and introduced by John Corcoran.

Tennant, Neil. 2002. "Deflationism and the Gödel-Phenomena." To appear in *Mind.*

Thomason, Richmond H. 1980. "A Note on Syntactical Treatments of Modality." *Synthese* 44:391–95.

Troelstra, Anne S. and Helmut Schwichtenberg. 1997. *Basic Proof Theory.* Cambridge Tracts in Theoretical Computer Science no. 43. Cambridge: Cambridge University Press.

Troelstra, A[nne] S. and D[irk] van Dalen. 1988. *Constructivism in Mathematics: An Introduction,* 2 vols. Studies in Logic and the Foundations of Mathematics no. 121 and 123. Amsterdam, New York, Oxford and Tokyo: North-Holland.

Unger, Peter. 1979. "I Do Not Exist." In *Perception and Identity: Essays Presented to A. J. Ayer with his Replies to them*, edited by G[raham] F. Macdonald, 235–51. London and Basingstoke/Ithaca (New York): Macmillan/Cornell University Press.

———. 1980. "The Problem of the Many." *Midwest Studies in Philosophy* 5:411–67.

van Fraassen, Bas C. 1966. "Singular Terms, Truth-Value Gaps, and Free Logic." *Journal of Philosophy* 63 (17): 481–95 (September).

Visser, Albert. 1989. "Semantics and the Liar Paradox." In *Topics in the Philosophy of Language*, edited by D[ov] M. Gabbay and F[ranz] Guenthner, Volume 4 of *Handbook of Philosophical Logic*, 617–706. Dordrecht: Reidel.

von Neumann, J[ohn]. 1925. "Eine Axiomatisierung der Mengenlehre." *Journal für die reine und angewandte Mathematik* 154 (4): 219–40 (October). English translation by Stefan Bauer-Mengelberg and Dagfinn Føllesdal as "An axiomatization of set theory" in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, edited by Jean van Heijenoort, pages 393–413, Cambridge (Mass.): Harvard University Press, 1967.

Williams, Michael. 1986. "Do We (Epistemologists) Need A Theory of Truth?" *Philosophical Topics* 14 (1): 223–42 (Spring).

Yablo, Stephen. 1985. "Truth and Reflection." *Journal of Philosophical Logic* 14:297–349.