Truth and Finite Conjunction

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Abstract

This note is a critical response to Kentaro Fujimoto's new conservativeness argument about truth, which centres on the notion of finite conjunction. We argue that Fujimoto's arguments turn on a specific way of formalising the notions of finite collection and finite conjunction in first-order logic. In particular, by instead formalising these concepts in a natural way in set theory or in second-order logic, Fujimoto's new conservativeness argument can be resisted.

1 Deflationism and conservativeness

According to the central claim of *deflationism about truth*, the concept of truth is a 'light', 'insubstantial' notion. This raises the question what kind of notion truth is instead. Many deflationists take truth somehow to be a *quasi-logical* notion. Just as it makes no sense to ask what the essence of 'and' is, so it makes no sense to ask what the essence of truth is. Rather, like 'and', truth is a notion that helps us reason well.

The claim that truth is an insubstantial notion is rather vague; it can be made clearer in several ways. One way in which the claim that truth is an insubstantial notion can be made more precise is in terms of *proof theoretic conservativeness*. On this precisification, a notion of truth that is governed by a set of axioms Tr is deflationist if for any theory S that does not explicitly involve the notion of truth, the extension of S with the axioms of Tr—call this theory $Tr[S]$—is proof theoretically conservative over S. Here $Tr[S]$ is said to be proof theoretically conservative over S if and only if $Tr[S]$ proves no *new* theorems in the language of S: if a theorem in the language of S is provable in $Tr[S]$, then it is already provable in S alone (i.e., without making use of the concept of truth).

Conservativeness deflationism has been implicitly or explicitly endorsed by several authors. Horsten and Leigh, for instance, defend a conservative type-free disquotational truth theory as our fundamental theory of truth, and thus indirectly endorse conservativeness deflationism.[1] Waxman explicitly endorses conservativeness deflationism in the following passage:

> Is there any reasonable scope for denying that a deflationist theory of truth must be conservative? [...] The transition [from the claim that truth is insubstantial to the claim that truth is conservative] has considerable intuitive force, for it seems extremely uncomfortable to maintain that truth is an insubstantial or non-robust property if the addition of truth principles leads one to rule out what were previously considered to be live possibilities concerning a (truth-free) subject matter. Perhaps the best way of understanding the transition is as a proposed explication: the informal notion of metaphysical insubstantiality is to be (possibly partially) explicated in terms of the formal criterion of conservativeness. It is striking, and a mark in favour of the plausibility of this understanding, that the conservativeness requirement has attracted considerable support among deflationists themselves. [Waxman 2017, p. 445–446]

Conservativeness deflationism can also be made more precise in terms of other concepts, such as semantic conservativeness. However, the proof theoretic explication is often presupposed in the debate about the viability of deflationism about truth. It plays a central role in the so-called *conservativeness argument* against deflationism of truth, which goes as follows. A powerful and seemingly unobjectionable theory of truth is the compositional truth theory $CT[S]$, according to which the truth predicate commutes with the first-order logical connectives. It can be shown that under fairly general conditions, $CT[S]$ is *not* proof theoretically conservative over a background theory S when the schematic commitments of S (such as its logical commitments, and its commitment to the principle of mathematical induction) are taken in an open-ended way rather than restricted to the language in which S is formulated. *Therefore*—or so the argument goes—truth is a substantial notion, and deflationism about truth is false.

Even given the proof theoretic explication of deflationism about truth, the conservativeness argument is controversial. Perhaps the most influential objection against it was raised by Field in [Field 1999].[2] He argues that extending mathematical induction so as to include predicates containing the truth predicate amounts to a mathematical strengthening of the background theory. If this is right, then the conservativeness argument is blocked. Let us denote the result of adding the compositional axioms for truth *without extending the schematic mathematical axioms* of a background theory S as $CT^-[S]$ (while continuing to reserve the designation $CT[S]$ as the result of extending S with the compositional truth axioms *and* extending all its schemes to allow instances of the truth predicate). Then $CT^-[S]$ is proof theoretically conservative over S, because inductive arguments concerning formulas including the truth predicate then cannot be carried out in $CT^-[S]$.

In [Fujimoto 2022], Fujimoto formulates and discusses a *new conservativeness argument*. Like the 'old' conservativeness argument, the new argument also intends to establish the proof theoretic non-conservativeness of truth, and thereby to refute deflationism about truth. But it is intended to be convincing even to those who accept Field's critique of the old conservativeness argument.

The basic idea of the new conservativeness argument is that $CT^-[S]$ fails fully to capture the compositionality of the truth predicate. In particular, $CT^-[S]$ does not prove that any *arbitrary finite* conjunction of (truth-free) statements is true if and only if every one of these statements are true. This property of truth is called the principle of *conjunctive correctness*. Fujimoto produces informal arguments the validity of which turns on instances of the principle of conservative correctness. Thus, Fujimoto argues, the truth theory $CT^{-}[S]$ should be extended by the principle of conjunctive correctness. But it has been shown in recent years that if $CT^{-}[S]$ is extended by a particular formalisation of conjunctive correctness, then the resulting theory $CT^{cc}[S]$ is under fairly general conditions proof theoretical non-conservative over S *even if no predicates containing the truth predicate are allowed in arguments by mathematical induction*. Thus deflationism of truth fails even if Field's critique of the 'old' conservativeness argument is correct.

We will see how Fujimoto's argument turns out to be quite sensitive to the way in which conservative correctness is formally expressed. Fujimoto formalises conjunctive correctness in a first-order setting, in terms of an arithmetical coding of finite sets and an elementhood relation on finite sets. If instead the notion of finite set is formalised in a natural way in second-order logic or in set theory, then the resulting truth theory is proof theoretically conservative.

We will argue that the straightforward formalisation of finiteness in second-order logic has an intuitive appeal that Fujimoto's formalisation involving coded expression of finite sets in first-order arithmetic lacks. One can argue about the advantages and disadvantages of formalising conjunctive correctness in first-order arithmetic, set theory, or second-order arithmetic. But it should not be a commitment *of truth theory* that the concepts of finite set and of finite conjunction are expressed in one particular setting. Therefore Fujimoto's new conservativeness argument should not sway even the philosophers who accept the proof theoretic explication of deflationism about truth.

2 Blind deduction

Following Fujimoto, we work in a typed setting, so that the discussion is unaffected by complications induced by self-applicable truth. Moreover, as is often done, for concreteness we mostly take, in what follows, the background theory S to be PA. But the arguments that we will give are intended to apply more generally to compositional truth applied to arbitrary background theories, as long as they are mathematically sufficiently strong. Indeed, in section 4.1 we briefly discuss the situation where not first order arithmetic, but rather first order set theory, is taken as background theory. Moreover, we will consider the prospects of appealing to second-order resources in the formalisation of arguments that involve the concept of arbitrary finite conjunction.

Basic truth principles are needed to explain why certain apparently valid informal arguments using the concept of truth indeed are valid. Fujimoto argues that for truth theories adequately to fulfil this task, they must at least contain the compositional truth axioms [Fujimoto 2022, section 3]. In other words, he argues that the truth theory CT^{-} should be taken for granted. We do not challenge this assumption in this note.

Truth is used in what Fujimoto calls *blind deductions*, which are "deductive arguments about the truth of some sentences by analysing and manipulating their logico-linguistic structure without explicitly specifying what these sentences are" [Fujimoto 2022, p. 137].

Examples of blind deductions play a key role in Fujimoto's new conservativeness argument. In particular, Fujimoto makes use in his new conservativeness argument of the following three pieces of informal reasoning:

ARGUMENT 1 [Fujimoto 2022, p. 147]

P1 All the axioms of *PA* are true.

P2 Amy wrote down some (finitely many) axioms of *PA* in her notebook.

P3 If what Amy wrote down in her notebook is all true, then Cathy's conjecture is true.

P4 Cathy made exactly one conjecture.

C1 Cathy's conjecture is true.

ARGUMENT 2 [Fujimoto 2022, p. 149]

P1,2 (The first two premises of Argument 1.)

P5 Beth denied one of the sentences that Amy wrote in her notebook.

P6 If Cathy's conjecture is true, then Beth's claim is true.

P7 Beth made exactly one claim and Cathy made exactly one conjecture.

C2 If what Amy wrote in her notebook is all true, then Cathy's conjecture is not true.

ARGUMENT 3 [Fujimoto 2022, p. 151]

P1,2,7 (Premises 1, 2, and 7.)

P8 Beth claimed *the conjunction of* what Amy wrote in her notebook implies Cathy's conjecture

C3 If Beth's claim is true, then Cathy's conjecture is true.

Observe that Argument 3 can be seen as a "variant" of Argument 1.

Let us violate Russellian strictures about the fomalisation of descriptions slightly by formalising 'Cathy's conjecture' as a (first-order) individual constant a and 'Beth's claim' as a (first-order) individual constant b. This simplifies matters—since we can then ignore premises P4 and P7,—without affecting the structure of the argument (as the reader can readily verify).

Fujimoto claims that all three Arguments are intuitively valid [Fujimoto 2022, p. 147, p. 149, p. 150], and so do we. If truth is a quasi-logical notion, then an adequate axiomatic theory must bear this out, by being such that from correct formalisations of the premises, the conclusions can be *derived* using truth axioms.

3 From blind deduction to conjunctive correctness

The concept of finiteness seems to play a role in all three arguments. In fact, we will see that it is not clear that the concept of finiteness plays an essential role in the first two arguments; but is a crucial ingredient in Argument 3.

The following is a first-order way of making sense of the concept of finiteness that is at play. There is an arithmetical expression $\underset{\cdot}{\varepsilon}$ belonging to the language of first-order Peano Arithmetic ($\mathcal{L}_{PA}$) such that for every finite set of (codes of) sentences X, there is an arithmetical code c of X such that, for all natural numbers n,

$$n \in X \Leftrightarrow n \underset{\cdot}{\epsilon} c.$$

Moreover, PA proves (a coded version of) comprehension for finite predicates [Fujimoto 2022, p. 145]:

Definition 1 (FC) $\Phi(x)$ *has a finite extension, i.e.,* $\exists n \forall x(\Phi(x) \rightarrow x \leq n)$ *iff* $\exists c \forall x(x \underset{\cdot}{\epsilon} c \leftrightarrow \Phi(x))$.

In addition, there is an arithmetically definable function $\underset{\cdot}{\bigwedge}$ that transforms a code c of a finite set of sentences into the conjunction of these sentences (with a given bracketing convention). This then gives us a notion of *blind conjunction* ($\underset{\cdot}{\bigwedge} x$).

Now for any given a finite collection X of sentences, Fujimoto claims that the truth of the blind conjunction of the X-es should be formalised as

$$T(\underset{\cdot}{\bigwedge} c),$$

with c the code of X, and T a primitive truth predicate [Fujimoto 2022, p. 146].

Then the following principle, which is known as the axiom of *Conjunctive Correctness*, can be formulated:

Axiom 1 (CC)

$$\forall c : (\forall x(x \underset{\cdot}{\epsilon} c \rightarrow x \in \mathcal{L}_{PA})) \rightarrow ((\forall x(x \underset{\cdot}{\epsilon} c \rightarrow T(x))) \leftrightarrow T(\underset{\cdot}{\bigwedge} c)).$$

The version of CC with the consequent restricted to a left-to-right implication is known as CCintro. The version of CC with the consequent restricted to a right-to-left implication is known as CCelim.

Define $CT^{cc}[PA]$ as the theory resulting from adding the axiom CC to $CT^{-}[PA]$. Enayat and Pakhomov proved the following surprising theorem:

Theorem 1 *[Enayat & Pakhomov 2019]*

$$CT^{cc}[PA] \vdash CT_0[PA],$$

where $CT_0[PA]$ is like $CT^-[PA]$, except that the induction axioms for *quantifier-free* atomic formulas *that may contain occurrences of the truth predicate* are also included. Now Wcisło and Łełyk have shown that $CT_0[PA]$ is arithmetically non-conservative over PA [Wcisło & Łełyk 2017]. So this means that $CT^{cc}[PA]$ is also arithmetically non-conservative over PA.

With all this in place, Fujimoto argues that the conclusions of Argument 1 and Argument 2 can be derived from their premises *only if* CC holds. More specifically, in the context of $CT^-[PA]$ Argument 1 is a derivable argument scheme only if CCintro holds, and Argument 2 is a derivable argument scheme only if CCelim holds [Fujimoto 2022, section 4]. We do not rehearse his argument here, but merely stress that his argument heavily depends on formalising these arguments using the machinery of coding finite sequences in first-order arithmetic in the way described above. Since in the context of $CT^-[PA]$, CC is arithmetically non-conservative over the background arithmetical theory PA, Fujimoto concludes that truth is non-conservative.

4 Truth, finiteness, and second-order logic

We now turn to the evaluation of Fujimoto's *new* conservativeness argument. All three Arguments are intended to indicate that conjunctive correctness should be added to $CT^-[PA]$ as a fundamental truth axiom, and all three Arguments are somehow connected with the notions of finiteness. We accept that a form of conjunctive correctness needs to be provable from our basic principles governing the notion of truth. In the following, we critically evaluate the role and formal treatment of finiteness in Fujimoto's three Arguments.

4.1 In set theory

The notion of finiteness is of course straightforwardly expressible in the language of set theory ($\mathcal{L}_{ZFC}$). So suppose we take first-order ZFC as our background theory, and—like in the arithmetical case—add compositional truth axioms to it, but be careful *not* to allow the truth predicate to occur in the separation and replacement schemes. Call the resulting theory $CT^-[ZFC]$. Then we can define a natural (coding-free) notion of corrective correctness in the following manner:

Definition 2 (CC^{set}) $\forall x : [\mid x \mid < \omega \wedge \forall y \in x : y \in \mathcal{L}_{ZFC}] \rightarrow [T(conj(x)) \leftrightarrow \forall y \in x : Ty]$.

It is clear that in the theory $CT^{-}[ZFC] + CC^{set}$, the obvious formalisations of the three Arguments can be proved.

However, it follows from an argument by Fujimoto[3] that :

Theorem 2 *$CT^{-}[ZFC] + CC^{set}$ is conservative over ZFC for the language of set theory.*

Thus, as Fujimoto himself notes ([Fujimoto 2022, p. 155]), Fujimoto's new conservativeness argument does not go through in this setting.

What is wrong with formalising the three Arguments in the setting of set theory? One possible worry might be that the ontological commitments of set theory far outstrip the ontological commitments of the three Arguments. It might seem, in other words, that the price for ideological conservativeness is ontological non-conservativeness. For this reason, we shall now attempt to show that even in a setting that is ontologically conservative over first-order arithmetic, the three Arguments do not force non-conservativeness of truth upon us.

4.2 The first two arguments

The qualification "finitely many" is in brackets in Argument 1 and implicitly assumed in Argument 2 (witness Fujimoto's formalisation of Argument 2 on [Fujimoto 2022, p. 149]), so it is somewhat ambiguous whether it belongs to the argument. If we ignore the qualification "finitely many" in our formalisation (and therefore do not need the first-order machinery of coded finite sets at all), then the validity of Arguments 1 and 2 can be witnessed in the background theory alone or in $CT^{-}[PA]$. So, in that case, non-conservativeness does not follow from these arguments. Let us formalise Argument 1 in this way, where the predicate N formalises 'is written down by Amy' (and, as said before, a is an individual constant referring to Cathy's conjecture):

1. $\forall x : AxPA(x) \rightarrow Tx$
2. $\forall x : N(x) \rightarrow AxPA(x)$
3. $(\forall x : N(x) \rightarrow T(x)) \rightarrow T(a)$

4. $T(a)$

It is immediate that the last sentence is derivable from the previous ones. Indeed, truth laws play no role in this derivation. Thus the parenthetical finiteness assumption appears to be a red herring. Moreover, we do not even need the truth laws of $CT^{-}[PA]$ in this derivation. This depends on formalising 'A implies B' as $T(A) \rightarrow T(B)$.[4] Alternatively, one could formalise 'A implies B' as $T(A \rightarrow B)$. Then some of the compositional truth axioms of $CT^{-}[PA]$ play a role in the derivation. In either case, on this reading of Argument 1, we do not obtain proof theoretic non-conservativeness.[5]

As is well-known, the notion of finiteness can explicitly be *defined* in a simple and natural way in second-order logic. Moreover, since second-order logic can be interpreted in an *ontologically* non-inflationary way as plural logic,[6] we take it in principle to be philosophically unobjectionable to make use of second-order logic for purposes of formalisation of natural language arguments.

If we do build the parenthetical finiteness claims into our formalisation, but formalise finiteness in a second-order setting, then the conclusions of Fujimoto's first two arguments again follow from their premises very directly. Let $FIN(X)$ be a standard second-order definition of what it means for X to be finite. Then Argument 1, for instance, can be formalised in the language of second-order arithmetic as follows:[7]

1. $\forall x : AxPA(x) \rightarrow Tx$
2. $\exists X[FIN(X) \wedge \forall y : N(y) \leftrightarrow (y \in X \wedge AxPA(y))]$
3. $(\forall x : N(x) \rightarrow T(x)) \rightarrow T(a)$
4. $T(a)$

For the last sentence to be derivable from the premises, it suffices to derive $\forall x : N(x) \rightarrow T(x)$ from the first two premises. But this can easily done using just the normal existential generalisation / instantiation rules for second-order logic,[8] and without using truth laws. Again, the bit about finiteness in the formalisation plays no active role in the derivation: it is a red herring. Note that in particular, therefore, no use is made of any kind of second-order comprehension (or mathematical induction) in this derivation. This means that the whole derivation can easily take place in a second-order theory such as ACA_0, which is first-order conservative over PA (and even this is overkill).[9]

4.3 The third argument

Fujimoto's Argument 3 is subtle. According to P8, Beth does not make a claim concerning any *specific* conjunction of statements: Beth makes a *de dicto* rather than a *de re* claim. This is the reason why the concept of *arbitrary* finite conjunction is needed to formalise the Argument.[10] Nonetheless, as we shall now argue, we do not need to appeal to a non-conservative extension of $CT^{-}[PA]$ to derive its conclusion from its premises.

First, we show how being a finite conjunction can be defined in a natural way in second-order logic. We work in the language of relational second-order logic over the language of arithmetic. In addition to the theorems of PA, we assume that (monadic and relational) second-order comprehension holds for all formulas in the language: we can thus comprehend on formulas involving arbitrary arithmetic and relational second-order resources.[11] On the other hand, the *second-order* induction axiom is not assumed in our second-order framework.

We start by being precise about what we will mean by "finite" in what follows:

Definition 3 *We say that a set X is* **finite** *if it is finitely enumerable. More precisely, we say that X is finite when there is a well-order R on X which is reverse well-founded. (Equivalently, there is a well-order R on X such that: X has an R-last element, and every element of X is either the R-least element or an R-immediate successor of some other element).*

Lemma 1 *If X is finite, then $<$ is a reverse well-founded well-order on X.*

Proof. *Let R witness the fact that X is finite. Trivially, $<$ is a linear order on X. So, suppose it is not well-founded and let $Y \subseteq X$ have no $<$-least element. We can then define, by recursion on R, a functional relation R' such that (i) R''s domain is X, (ii) if x is the R-least element of X, then $R'(x, y)$ where $y \in Y$ is some arbitrarily chosen object, and (iii) if x is the immediate R-successor of y and $R'(y, z)$, then $R'(x, w)$ where w is the R-least element of Y $<$-below z. If x is the R-greatest element of X and $R'(x, y)$, then there is an element of Y $<$-below y and therefore not in the range of R'. So, R' codes a one-one function from X into* *one of its proper subconcepts, which is impossible.*[12] *The argument is similar if $<$ is not reverse well-founded.* ■

Next, we define the way in which a conjunction of a finite set of formulas is inductively built up:

Definition 4 *Say that R is a (canonical)* **conjunction sequence** *for a finite set of formulas X if (i) R is functional, (ii) R's domain is X, (iii) if x is the $<$-least member of X, then $R(x, x)$, and (iv) if $x \in X$ is the $<$-immediate successor in X of $y \in X$ and $R(y, z)$, then $R(x, w)$ where $w = z \wedge x$. (Since X is assumed to be a set of formulas, $z \wedge x$ is well-defined for $z, x \in X$.)*

Next, it can be shown that all finite sets of formulas have conjunction sequences:

Lemma 2 *If X is a finite set of formulas, then it has a unique (up to extension) conjunction sequence.*

Proof. *By Lemma 1, $<$ is a reverse well-founded well-order on X. We prove the existence of unique conjunction sequences by induction on $<$ over X.*[13] *Clearly, there is such a sequence for the $<$-least element of X and its $<$-predecessors in X. So, suppose R is a conjunction sequence for $x \in X$ and its $<$-predecessors in X. Let $y \in X$ be x's immediate $<$-successor in X, and let z be such that $R(x, z)$. Then $R' = R \cup \langle y, z \wedge y \rangle$ is a conjunction sequence for y and its $<$-predecessors in X. Moreover, since R is unique up to extension, so too is R'.* ■

Definition 5 *When X is a finite set of formulas, let $CONJ(X, x)$ abbreviate the claim that any (equivalently: some) conjunction sequence R for X is such that $R(y, x)$, where y is the $<$-greatest element of X. Let $FIN(X)$ abbreviate the claim that X is a finite set of formulas.*

Theorem 3 $\forall X(FIN(X) \rightarrow \exists! x\, CONJ(X, x))$.

Proof. *Trivial from Lemma 2.* ■

If we had an axiom of Global Well-Ordering, we could use Dedekind-finiteness as our notion of finiteness. In the absence of such an axiom, enumerable finiteness (Definition 3) is often taken to be the right notion. However, our argument above with the enumeration notion of finiteness might be regarded as preferable over the strategy with Dedekind-finiteness plus

Global Wellordering instead. This is because some may see Global Wellordering not as a logical but as a mathematical principle, and would then argue that through assuming a Global Wellordering, non-conservativeness enters through the back door.

To some extent, our argument is in the spirit of recent projects that attempt to secure conservativeness by separating syntax from subject matter (see, for example, [Leigh & Nicolai 2013]). However, flat footedly doing that in response to Fujimoto's argument would be ineffective. Finiteness is, arguably, an arithmetical property more so than a syntactic one. In contrast, our notion of finiteness is as natural as the arithmetical one.

It follows from Theorem 3 that we can treat CONJ as a function symbol: with mild abuse of language, for any set A, let $CONJ(A)$ be the conjunction of the elements of A if A is a finite set of sentences of $\mathcal{L}_{PA}$ (and a number that is not the code of a sentence otherwise). Now we can formalise Argument 3 as follows:

1. $\forall x : AxPA(x) \rightarrow Tx$
2. $\exists X : FIN(X) \wedge \forall y : N(y) \leftrightarrow (y \in X \wedge AxPA(y))$
3. $T(CONJ(N)) \rightarrow Ta$
4. $T(a)$

In order for the conclusion to be provable from the premises, we need the following second-order version of conjunctive correctness:

Axiom 2 (CC^2)

$$\forall N : FIN(N) \rightarrow (\forall y(Ny \rightarrow Ty) \leftrightarrow T(CONJ(N))$$

The principle CC^2 is a very natural way of expressing that an arbitrary finite conjunction is true iff all its conjuncts are true. We now show that CC^2 can be conservatively added to CT^-. Let SOL be any reasonable system of second-order logic. It may contain full or only restricted second-order comprehension, and second-order choice principles. Then we have:

Proposition 1 *The theory $CT^-[PA] + SOL + CC^2$ is first-order arithmetically conservative over PA.*

Proof. *We show that any model of $CT^-[PA]$ can be expanded to a model of $CT^-[PA] + SOL + CC^2$. Since $CT^-[PA]$ is first order arithmetically conservative over PA, this establishes the conclusion.*

*Take any first-order model $\mathcal{M}$ such that $\mathcal{M} \models CT^-[PA]$. Take the **standard** second-order expansion $\mathcal{M}'$ of $\mathcal{M}$ to the language of second-order arithmetic, i.e., $\mathcal{M}'$ is like $\mathcal{M}$ except that it also interprets second-order quantifiers, and it takes the second-order quantifiers to range over **all** subsets of the (first-order) domain of $\mathcal{M}$ (standard or non-standard). We will show that $\mathcal{M}'$ is the expansion that we are looking for.*

Because $\mathcal{M}'$ interprets the second-order quantifiers in a standard way, it makes SOL true. So it suffices to verify that $\mathcal{M}'$ also makes CC^2 true. Again because it is standard for the second-order quantifiers, we have:

$$\mathcal{M}' \models FIN(X) \Leftrightarrow X \text{ is a finite subset of the domain of } \mathcal{M}'.$$

Now (speaking somewhat informally), take any X such that $\mathcal{M}' \models FIN(X)$. Then X really is finite: say it consists of n elements $y_1, \ldots, y_n$ of the domain.

(a) *Suppose that for every $i < n$, we have $\mathcal{M}' \models T(y_i)$. We know that $\mathcal{M}' \models CT^-$, so (by a simple inductive argument in the metalanguage) we see that $\mathcal{M}'$ satisfies*

$$(T(y_1) \wedge \ldots \wedge T(y_n)) \rightarrow T(y_1 \wedge \ldots \wedge y_n),$$

and moreover we have $\mathcal{M}' \models CONJ(N) = y_1 \wedge \ldots \wedge y_n$. So we have $\mathcal{M}' \models T(CONJ(N))$.

(b) *Conversely, we see in a similar way that if $\mathcal{M}' \models T(CONJ(N))$, then $\mathcal{M}' \models T(y_i)$ for each $i < n$.* ■

5 Concluding remarks

Field has argued that instances of induction that contain the truth predicate do not count as truth laws because mathematical induction is a mathematical property: it holds in virtue of the natural numbers rather than in virtue of truth.

Finiteness is also a mathematical property. We should be given the freedom to formalise in one of several acceptable ways. It is not the business of truth theory to prescribe how it should formally be expressed. In particular, *as far as truth theory goes*, it is permissible to treat it as a second-order concept. But we have seen that if we do this, then Fujimoto's non-conservativeness argument no longer goes through. This can be seen as an indication that in formalising finite correctness as Fujimoto does, using numerical codes of finite sets, one is injecting new *mathematical* content into the theory. Or, in Fieldian terms, the worry is that the principle CC may not be a purely *truth theoretical principle* after all.

In extending $CT^{-}[PA]$ to a second-order theory, we did not extend the first-order induction scheme of PA to the second-order mathematical induction *axiom*. Someone might object, however, that we *should* do this, and observe that this will result in a second-order theory that is *not* conservative over PA. However, this would amount to conceiving of mathematical induction in an open-ended way. This attitude is, as we have seen, exactly what Field criticised in his rejection of the 'old' conservativeness argument [Field 1999]. In other words, if induction is open-ended, then no *new* conservativeness argument is needed.

There is an interesting remaining question concerning extending $CT^{-}[PA]$ to a second-order setting, however. It might be argued that it is natural to add a compositional truth clause that says that truth also commutes with the second-order quantifiers. Moreover, since not all values of second-order variables have names in the language, it is then natural to switch to satisfaction clauses instead of truth clauses. Let the resulting theory be called $CT^{2-}[PA]$. Then the reasoning of the proof of Proposition 2 cannot be used to establish that the extension $CT^{2-}[PA] + SOL + CC^{2}$ is first-order arithmetically conservative over PA. Whether the resulting theory is conservative or not, appears to be an open problem.[14]

Notes

[1]See [Horsten & Leigh 2017].

[2]For a recent critical discussion conservativeness deflationism, see [Murzi & Rossi 2020].

[3]See [Fujimoto 2012, Theorem 20].

[4]Actually, it should probably rather be formalised as '*Necessarily*, if A is true, then B is true'. But, like Fujimoto, we ignore the modal aspect of implication in this note.

[5]A completely parallel analysis can be given of Argument 2. We leave this analysis to the reader.

[6]See for instance [Boolos 1984].

[7]Again we leave the completely analogous formalisation of Argument 2 to the reader.

[8]These rules are onobjectionable: they are completely parallel to the usual existential generalisation / instantiation rules of first-order logic.

[9]We will later see that comprehension *does* play a role in dealing with Argument 3.

[10]Otherwise, as an anonymous referee rightly observed, there would be no need to appeal to the concept of *arbitrary* finite conjunction in the formalisation of Argument 3.

[11]Alternatively, we could work in a monadic second-order logic and code relations as concepts of arithmetically coded ordered pairs.

[12]This is so because finite enumerability implies Dedekind finiteness even in the absence of a Global Wellordering principle.

[13]Notice that we're not doing induction on $<$ in general, but only on $<$ restricted to X. So, we do not need arithmetical induction on second-order formulas to carry it out. We do, however, use arithmetical facts about formulas in the induction: like, e.g. that $x \wedge y$ is well-defined for formulas x and y.

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