Ranking Functions, AGM Style

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Ranking functions, having their first appearance under the name „ordinale Konditionalfunktionen" in my Habilitationsschrift submitted in 1983, had several precursors of which I was only incompletely aware, among them Shackle's functions of potential surprise (see Shackle 1969), Rescher's plausibility indexing (see Rescher 1976), Adams' ε-semantics (see Adams 1975), Cohen's inductive probabilities (see Cohen 1977), Shafer's consonant belief functions (see Shafer 1976, ch. 10), and Ellis' rational belief systems (see Ellis 1979).[1] Concerning the actual genesis, however, their ancestor was Peter Gärdenfors' early work on belief revision (see Gärdenfors 1979, 1981).[2] This work inspired me enormously, perhaps because I found there the dynamical perspective to be most salient, and so I eventually came up with the ranking functions. To my surprise, however, belief revision theory and ranking theory went mainly separate ways. I am not sure about the reasons (I offer some speculations below), but in any case I believe that the separation is unnecessary.

This paper intends to narrow the gap from the side of ranking functions. It tries to do so in four parts. Section 1 starts by resuming the mutual criticisms. Section 2 briefly introduces ranking theory and how it reflects belief revision theory. Section 3 tries to overcome the main reservation concerning ranking functions by discussing

[1] One must bear in mind that the ideas of these authors are often much older then the references I have given suggest.

[2] There is a huge amount of further material which starts to develop in the seventies (and earlier) and which is closely, but not so intimately related as the theories mentioned: for instance the Chisholm-Pollock-Lehrer account of defeasible reasoning (see perhaps first of all Pollock 1990, though the three authors must be distinguished, of course) and in particular such theories as default logic, nonmonotonic reasoning, possibility theory etc. best surveyed perhaps in Gabbay et al. (1994a).

the extent to which their structure is reflected in changing beliefs. Section 4, finally, presents a representation of the intended kind.

My paper is far from giving a complete account of the relation between ranking and belief revision theory. My point is rather only a conceptual one – ranking functions can take AGM shape! –, and therefore I am happy with offering one plausible connection, leaving room for variations and improvements. All proofs of theorems and other observations in the paper are more or less on the level of exercises; I have omitted them.

1. Mutual criticisms

My two main criticisms of belief revision theory have stayed the same in the last 15 years. They were directed against Gärdenfors' early papers, but as far as I see they still apply to the AGM theory (cf. Alchourrón et al. 1985) and to the more recent developments (cf. Gärdenfors and Rott 1995, Rott 1999a, or Hansson 1998, 1999).

The first point is the well known problem of iterated belief revision. One must be aware that this is not just some important problem; it is *vital* to belief revision theory. As long as it is not solved, belief revision theory does not deserve its name, since it does not specify a full dynamics (or kinematics) of belief. Ranking functions have been my answer to this problem (see Spohn 1988). The problem has also been attended to in the AGM framework[3], but neither has the problem received the central role it deserves, nor can I find the existing proposals convincing. Ranking functions still strike me as the more elegant and powerful solution; however, I do not want to engage now into a detailed argument about this.[4]

The second point is that belief revision theory does not have an adequate notion of doxastic dependence and independence, i.e. of irrelevance and positive and negative relevance. The most natural notion of independence in belief revision theory is the following: ψ is independent of ϕ relative to the belief set K and the revision operator * if and only if revision by ϕ and by $\neg\phi$ does not affect the doxastic status

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[3] See Boutilier (1993, 1996), Darwiche and Pearl (1997), Hansson (1992, 1993), Lehmann (1995), Nayak (1994), Nayak et al. (1996), and Rott (1998, 1999b).

[4] I am surprised, however, to see that revision methods like that of Boutilier (1993) are still under discussion, although they were envisaged already in Spohn (1988, sect. 3) and found to be clearly inadequate.

of ψ as being believed or disbelieved or neither; i.e., if and only if it holds that $\psi \in K^*\phi$ iff $\psi \in K^*\neg\phi$ and $\neg\psi \in K^*\phi$ iff $\neg\psi \in K^*\neg\phi$. However, as I mentioned in Spohn (1988, p. 120; cf. also 1983, footnote 18), this is too much independence; it entails, for instance, that each sentence ψ believed in K is doxastically independent of each sentence ϕ neither believed nor disbelieved in K. Rather such a ϕ should possibly be positively (or negatively) relevant to such a ψ. For instance, I do not know whether the candidate speaks French (= ϕ) or not, but I believe (because of her other qualities) that she will get the job (= ψ). But since speaking French is an additional qualification for the job, ϕ is intuitively an additional reason or positively relevant for ψ. I am not aware of any more sophisticated accounts of doxastic dependence and independence within the AGM framework which would fare essentially better.

The point is a philosophically consequential one. Since Gettier (1963) an enormously rich, but also often frustrating discussion has developed about the nature of knowledge, justification and related matters. I find it obvious that belief revision theory is highly relevant to this discussion; the latter could indeed profit from the former in various respects. Therefore I applaud all attempts such as Rott (1999a) or Olsson (1999a,b) to realize this profit. However, I think that these attempts are severely handicapped from the outset by the lack of a workable notion of doxastic dependence and independence. By contrast, my favorite, and, as I believe, successful explication of the relation of being a reason is simply positive relevance (cf. Spohn 1983, 1997/98, 1999), provided it is based on an adequate account of relevance.

A related consequence is this: The theory of Bayesian nets (see, e.g., Pearl 1988 or Jensen 1996), the beauty and usefulness of which has been fully recognized in AI, but hardly in philosophical epistemology, entirely depends on the suitable properties of conditional independence which can be graphically represented by so-called d-separation. Probability theory yields such properties, ranking theory as well (cf. Spohn 1988, sect. 6, and 1994); hence, Bayesian nets can be developed in probabilistic as well as in ranking terms. But there seems no way to draw a connection to belief revision theory.

The two criticisms are clearly connected. I do not have a strict argument generally showing that the failure of iterated revision entails an inadequate grasp of independence. But the source of the two problems is the same in belief revision theory, and they vanish at once in ranking theory.

I am less certain about the converse criticism; there is apparently no published argument. From discussions, however, I infer that there are two main criticisms, one about the logical format and one from operationalism. Let me explain.

First, belief revision theory provides a logic of the operators studied by it, whereas ranking theory does not have the form of a logic at all. Thus, philosophers who are used to conceive of the theories at issue as a branch of philosophical logic are automatically deterred by ranking theory. In particular, a logical system is usually is ennobled by a completeness theorem. Belief revision theory indeed provides a number of such completeness results.[5] By contrast, it is already unclear what completeness could at all mean in ranking theory. This is a serious complaint. But it is, in a way, a variant of the second criticism, so that my response to the latter, which is the main business of this paper, will at the same time answer this complaint as well.

The second main criticism concerns the cardinal structure of ranks which makes people feel very uneasy; this structure appears to be a mysterious theoretical make-up lacking sober foundations. The basis of this uneasiness is, I suspect, a kind of operationalism: According to it, the primary data of epistemology are the beliefs of a given subject; they provide, as it were, the observational basis of any epistemological theory. They may be thought to do so either because beliefs are taken to be somehow introspectively accessible or because they are taken to have a clear and direct behavioral manifestation, say, in sincere speech. So, any theory which talks only about these primary data is fine; to assume less accessible structure is legitimate according to the operationalist attitude only if the structure is somehow uniquely reflected in the primary data; insofar it is not so reflected, it is an unsupported posit.

Belief revision theory perfectly meets these operationalist standards. In its axiomatic form it speaks only of the beliefs of a subject as they are before and after some doxastic operation like revision or contraction; the only hypothetical element is that all possible revisions etc. are considered which by far exceed the actual ones. In its semantic form it speaks of additional structures like choice functions (over models or in some sense maximal sets of sentences), entrenchment relations, etc., but then it goes on to show how these structures are uniquely manifested in the changing beliefs. This is precisely the point of the above-mentioned completeness results which turn into representation theorems under the present methodological

[5] This was, in a way, the essential achievement of Alchourrón et al. (1985), the birth of the AGM theory, which has found many variations since then.

perspective. No comparable achievements seem available for ranking functions, and as long as this is so, they cannot be accepted from such an operationalist point of view.

I have two responses to this criticism. First, I think that I was not unfair in characterizing the attitude in question as (kind of) operationalistic. So the obvious comment is that operationalism is dead since 30 or even 60 years, and for good reasons. Representation theorems are thus not necessary for rendering theoretical concepts meaningful; it would be wrong to discard a theory merely for want of a suitable representation in (quasi) observational terms. This is not to say that representation theorems are not useful. They are; they tell when our axiomatic and semantic intuitions coincide and mutually support each other. This is good to know, but it does not make them indispensible.

This is the offensive response which I take, however, to be unconvincing in the present context. So I add, secondly, a defensive response which is simply that it is quite straightforward to establish representation theorems for ranking functions as well; their theoretical structure is also uniquely reflected in changing beliefs. This is what I want to explain in the rest of the paper.

2. Ranking functions, revisions, and contractions

To this purpose we have to formally introduce ranking functions. To keep things simple I restrict everything to be finite. So let W be a finite set of basic possibilities (possible worlds) and let all subsets of W, denoted by A,B,C,D,E, and F with or without subsripts, be propositions. $\overline{A}$ denotes the complement $W \setminus A$ of A; often I abbreviate $A \cap B$ by AB.

Definition 1: κ is a *ranking function* (for W) iff κ assigns to each non-empty proposition a natural number as its rank such that for all A, B:

(a) either $\kappa(A) = 0$ or $\kappa(\overline{A}) = 0$ (or both),

(b) $\kappa(A \cup B) = \min\{\kappa(A), \kappa(B)\}$.

For $A \cap B \neq \emptyset$ the *conditional rank* of B given A is defined as:

(c) $\kappa(B \mid A) = \kappa(A \cap B) - \kappa(A)$.

Finally, define the *core* E of κ as

(d) $E = \{w \in W \mid \kappa(\{w\}) = 0\}$.

Ranks represent degrees of disbelief. Hence, A is not disbelieved in κ iff $\kappa(A) = 0$, disbelieved in κ iff $\kappa(A) > 0$, and believed in κ iff $\overline{A}$ is disbelieved in κ, i.e. iff $\kappa(\overline{A}) > 0$ or iff A is a superset of the core E. Thus, we may define the belief set K associated with κ as the set of propositions believed in κ, i.e. $K = \{A \mid E \subseteq A\}$. Obviously, there is a one-one-correspondence between belief sets and cores. I find it easier to proceed in terms of cores instead of belief sets. Due to (b), the rank of a proposition is the minimum of the ranks of its singleton subsets – a useful property I shall occasionally exploit.

The formal explication of doxastic dependence and independence will become important later on:

Definition 2: Let κ be a ranking function. Then A is $\begin{Bmatrix} \textit{positively relevant} \\ \textit{irrelevant} \\ \textit{negatively relevant} \end{Bmatrix}$ to B relative to κ iff $\begin{Bmatrix} \kappa(\overline{B} \mid A) > \kappa(\overline{B} \mid \overline{A}) \text{ or } \kappa(B \mid A) < \kappa(B \mid \overline{A}) \\ \kappa(\overline{B} \mid A) = \kappa(\overline{B} \mid \overline{A}) \text{ and } \kappa(B \mid A) = \kappa(B \mid \overline{A}) \\ \kappa(\overline{B} \mid A) < \kappa(\overline{B} \mid \overline{A}) \text{ or } \kappa(B \mid A) > \kappa(B \mid \overline{A}) \end{Bmatrix}$, i.e. iff $\begin{Bmatrix} \kappa(AB) + \kappa(\overline{A}\overline{B}) < \kappa(A\overline{B}) + \kappa(\overline{A}B) \\ \kappa(AB) + \kappa(\overline{A}\overline{B}) = \kappa(A\overline{B}) + \kappa(\overline{A}B) \\ \kappa(AB) + \kappa(\overline{A}\overline{B}) > \kappa(A\overline{B}) + \kappa(\overline{A}B) \end{Bmatrix}$. Moreover, A is $\begin{Bmatrix} \textit{positively relevant} \\ \textit{irrelevant} \\ \textit{negatively relevant} \end{Bmatrix}$ to B *given* C relative to κ iff these clauses hold for the corresponding ranks (additionally) conditional on C.

Of course, (conditional) irrelevance is the same as (conditional) independence. The equivalence of the two defining conditions is easily checked. The first condition directly expresses the intuitive meaning of relevance and irrelevance, whereas the second condition will later prove to be useful; it also shows the symmetry of irrelevance and positive and negative relevance. The inadequacies mentioned with respect to belief revision theory in the previous section are obviously avoided by this definition.

How to account for doxastic changes within this modelling? In Spohn (1988) I argued to conceive of it in close analogy to generalized probabilistic conditionalization invented by Jeffrey (1965, ch. 11). That is, if one is informed about A (and nothing else), then the ranks conditional on A and conditional on $\overline{A}$

should not change at all. This leaves little freedom; only the restriction of κ to A can be shifted relative to the restriction of κ to $\overline{A}$. In Spohn (1988) I argued further that there is no objective measure of how large the shift should be; rather, one and the same informational content can come in various strengths, and therefore the size of the shift should be a free parameter of the doxastic change. Hence, I defined:

Definition 3a: Let κ be a ranking function, A a non-empty proposition, and n a natural number. Then the *A,n-conditionalization* $\kappa_{A,n}$ of κ is defined by $\kappa_{A,n}(B) = \kappa(B|A)$ for $B \subseteq A$, $\kappa_{A,n}(B) = \kappa(B|\overline{A}) + n$ for $B \subseteq \overline{A}$, and $\kappa_{A,n}(B) = \min\{\kappa_{A,n}(AB), \kappa_{A,n}(\overline{A}B)\}$ for all other B.

Here, the parameter n characterizes the result of the doxastic change; in A,n-conditionalization the rank of A is shifted to 0 and the rank of $\overline{A}$ to n. As Shenoy (1991) has observed first and several others after him, it is in a way more natural to use the free parameter for measuring the size of the shift and not its result; only then it is appropriate to describe the parameter as characterizing solely the informational process. Then we get:

Definition 3b: Let κ be a ranking function, A a non-empty proposition, and n a natural number. Then the *A|n-conditionalization* $\kappa_{A|n}$ is defined by $\kappa_{A|n}(B) = \kappa(B) - m$ for $B \subseteq A$, where $m = \min\{\kappa(A), n\}$, $\kappa_{A|n}(B) = \kappa(B) + n - m$ for $B \subseteq \overline{A}$, and $\kappa_{A|n}(B) = \min\{\kappa_{A|n}(AB), \kappa_{A|n}(\overline{A}B)\}$ for all other B.

As intended, A improves here its rank in comparison to $\overline{A}$ by exactly n units. Of course, the two kinds of conditionalizations are interdefinable: $\kappa_{A,n} = \kappa_{A|m}$ with $m = \kappa(A) + n$, from which the converse relation may be inferred.

The essential point of these rules of doxastic change is that they can be unrestrictedly iterated; the result of conditionalizing a ranking function is a ranking function, which can be subject to further conditionalization.

Thereby we can immediately integrate belief revision theory into ranking theory, as I have mentioned in Spohn (1988, footnote 20).

Definition 4: Let κ be a ranking function with core E. Then define the *revision* $E*\langle A\rangle$ of E by $A \neq \emptyset$ relative to κ to be the core of $\kappa_{A,n}$ for some $n > 0$ and the *contraction* $E \div \langle A\rangle$ of E by $A \neq W$ relative to κ to be E, if $E \cap \overline{A} \neq \emptyset$, and to be the core of $\kappa_{A,0}$, if $E \subseteq A$.

It should be clear that this captures within ranking theory what we intuitively intend revision and contraction to be; it should also be clear that the result of a single revision does not depend on which $n > 0$ is chosen. The definition perfectly corresponds to revision and contraction as conceived in belief revision theory.

Theorem 1:

(a) $\emptyset \neq E^*\langle A\rangle \subseteq A$,

(b) if $E^*\langle A\rangle \cap B \neq \emptyset$, then $E^*\langle A\cap B\rangle = E^*\langle A\rangle \cap B$,

(c) if $E \cap \overline{A} \neq \emptyset$, then $E \div \langle A\rangle = E$,

(d) if $E \subseteq A$, then $E \div \langle A\rangle \cap A = E$,

(e) $E \div \langle A\cap B\rangle \subseteq E \div \langle A\rangle \cup E \div \langle B\rangle$,

(f) if $E \div \langle A\cap B\rangle \cap \overline{A} \neq \emptyset$, then $E \div \langle A\rangle \subseteq E \div \langle A\cap B\rangle$,

(g) $\mathrm{E}^*\langle A\rangle = E \div \langle A\rangle \cap A$,

(h) $E \div \langle A\rangle = E \cup E^*\langle \overline{A}\rangle$.

Compare now (a) and (b) of theorem 1 with the eight revision postulates of Gärdenfors (1988, sect. 3.3):

(K*1) $K^*\phi = \mathrm{Cn}(K^*\phi)$ (Closure),

(K*2) $\phi \in K^*\phi$ (Success),

(K*3) $K^*\phi \subseteq \mathrm{Cn}(K \cup \{\phi\})$ (Expansion),

(K*4) if $\neg\phi \notin K$, then $\mathrm{Cn}(K \cup\{\phi\}) \subseteq K^*\phi$ (Preservation),

(K*5) if $\mathrm{Cn}(\phi) \neq L$, then $K^*\phi \neq L$ (Consistency Preservation),

(K*6) if $\mathrm{Cn}(\phi) = \mathrm{Cn}(\psi)$, then $K^*\phi = K^*\psi$ (Intensionality)[6],

(K*7) $K^*(\phi\wedge\psi) \subseteq \mathrm{Cn}(K^*\phi \cup \{\psi\})$,

(K*8) if $\neg\psi \notin K^*\phi$, then $\mathrm{Cn}(K^*\phi \cup \{\psi\}) \subseteq K^*(\phi\wedge\psi)$.

(K*6) is implicit in my talking of propositions instead of sentences. (K*1) is contained in the correspondence between cores and belief sets. Given this correspondence, Theorem 1(a) is equivalent to (K*2) and (K*5), and Theorem 1(b) is equivalent to (K*7) and (K*8). (K*3) and (K*4) are entailed by (K*7) and (K*8), anyway. So, (a) and (b) of Theorem 1 are indeed equivalent to these revision postulates.

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[6] In the belief revision literature this property is called „extensionality", but there is an older tradition according to which (K*6) says that the belief revision operator is intensional.

Compare further Theorem 1, (c)-(f), with the eight contraction postulates of Gärdenfors (1988, sect. 3.4):

(K÷1)	$K \div \phi = \mathrm{Cn}(K \div \phi)$	(Closure),
(K÷2)	$K \div \phi \subseteq K$	(Inclusion),
(K÷3)	if $\phi \notin K$, then $K \subseteq K \div \phi$	(Vacuity),
(K÷4)	if $\phi \in K \div \phi$, then $\phi \in \mathrm{Cn}(\varnothing)$	(Success),
(K÷5)	$K \subseteq \mathrm{Cn}(K \div \phi \cup \{\phi\})$	(Recovery),
(K÷6)	if $\mathrm{Cn}(\phi) = \mathrm{Cn}(\psi)$, then $K \div \phi = K \div \psi$	(Intensionality),
(K÷7)	$K \div \phi \cap K \div \psi \subseteq K \div (\phi \wedge \psi)$,	
(K÷8)	if $\phi \notin K \div (\phi \wedge \psi)$, then $K \div (\phi \wedge \psi) \subseteq K \div \phi$.	

Again (K÷1) and (K÷6) are implicit in my framework. And due to the correspondence between cores and belief sets, (K÷2) - (K÷5) are equivalent to theorem 1, (c) and (d); and (e) and (f) translate (K÷7) and (K÷8). So, again, (c)-(f) of Theorem 1 are equivalent to these contraction postulates.

Finally, (g) of theorem 1 is obviously the Levi Identity, and (h) is the Harper Identity.

Indeed, theorem 1 is all that follows from the definition of revision and contraction in terms of ranking functions; no stronger properties of * and ÷ can be derived. This is an ambiguous assertion in view of the extensive discussion of the Gärdenfors postulates and many variants of them. It can either be used as confirmation of these postulates, as I tend to do. Or it can conversely be viewed as disconfirmation of ranking theory, if one thinks that these postulates are the wrong ones.

How do we know that Theorem 1 is complete? This becomes clear when one realizes that a ranking function κ embodies an epistemic entrenchment relation: B is at most as entrenched as A relative to κ iff $\overline{B}$ is at least as strongly disbelieved in κ as $\overline{A}$, i.e. iff $\kappa(\overline{B}) \geq \kappa(\overline{A})$. This entrenchment relation shows up in contractions: B is at most as entrenched as A iff $B \notin E \div \langle AB \rangle$. And it has just the properties which are revealed completely in the contraction postulates (K÷1) - (K÷8).[7] Moreover, it is obvious from Definition 4 that the revisions and contractions prescribed by a ranking function κ depend exclusively on the entrenchment relation entailed by it. Hence, theorem 1 is indeed complete.

[7] For all this see Gärdenfors (1988, sect. 4.6).

3. Ranking functions and iterated revisions and contractions

All this is well known for more than ten years.[8] So far, hence, we have arrived at the result that the ordering of disbelief entailed by a ranking function κ is uniquely reflected in single revisions and contractions as characterized in theorem 1 via the equivalence:

(Ord) $\kappa(A) \leq \kappa(B)$ iff $\overline{B} \notin E \div \langle \overline{A}\,\overline{B} \rangle$.

This amounts, conversely, to the negative fact that a ranking function itself cannot completely show up in single revisions and contractions, simply because many different gradings of disbelief result in the same order of disbelief. So the further strategy is clear: if we want to find out about a complete manifestation of ranking functions in changing beliefs, we have to look at iterated revisions and contractions. This is no surprise; after all, this is the use ranking functions were designed for, hence they should prove in this use. Of course, we thereby enter a large field of inquiry; one may think of many desirable or undesirable properties of such iterations. It is certainly beyond my power to provide a full investigation of this field; so I shall be content with offering a few plausible paths, since they are sufficient to establish my point.

Let me start with a very trivial observation. Revisions and contractions relative to ranking functions were based in definition 4 on the conditionalization explained in definition 3a. Of course, the other conditionalization gives rise to belief change as well.

Definition 5: Let κ be a ranking function with core E. Then define the (*minimal*) *enhancement* $E^{\#}\langle A \rangle$ of $A \neq \emptyset$ in E relative to κ to be the core of $\kappa_{A|1}$.

By a minimal enhancement A gets minimally better entrenched or, respectively, $\overline{A}$ gets minimally worse entrenched. Thus, enhancement is not revision; if $\kappa(A) > 1$ or $\kappa(\overline{A}) > 0$, $E^{\#}\langle A \rangle = E$, i.e. nothing changes on the surface. If $\kappa(A) = 1$, the

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[8] Actually, the basic facts are already contained, in somewhat different, though entirely differently interpreted terms, in Lewis (1973).

enhancement of A is, in effect, a contraction by $\overline{A}$; and if $\kappa(A) = \kappa(\overline{A}) = 0$, the enhancement of A is an expansion by A.

Enhancements can obviously be iterated in the same way as conditionalization. Hence define $E^{\#}\langle A_1,...,A_{n+1}\rangle = E^{\#}\langle A_1,...,A_n\rangle^{\#}\langle A_{n+1}\rangle$. Moreover, a proposition can be enhanced several times. Hence define $E^{\#n}\langle A\rangle = E^{\#}\langle A,...,A\rangle$ (n times). Now the trivial observation is that the grading of disbelief is uniquely reflected in iterated enhancements:

(Grad) $\kappa(A) = n > 0$ iff $E^{\#n\text{-}1}\langle A\rangle = E$ and $E \subset E^{\#n}\langle A\rangle$, and $\kappa(A) = \kappa(\overline{A}) = 0$
iff $E^{\#}\langle A\rangle = E \cap A$.

Furthermore, I see no principal difficulty in giving a complete „behavioral" description of iterated enhancements.

So, this already solves our problem. It is a disappointing solution, of course. The measuring rod enhancements provide is certainly unacceptable from the point of view of belief revision theory, since single enhancements have usually no „behavioral" consequences at all; only the appropriate number of enhancements has.

Let us hence look for a more convincing solution. One remark, though: Intuitively, enhancements make good sense. The following story is not unusual: I strongly disbelieve A. Now one source tells me that A. This cannot dispel my disbelief. The next source also affirms that A. I am still reluctant to give up my disbelief. But there comes the point where the number of affirmations outweighs my disbelief, provided they are independent and accumulate enhancements. Hence, epistemology is well advised to account for such cases. Probability theory is able to do so (if we neglect the problem of how at all to give a probabilistic account of belief and disbelief), ranking theory as well, but belief revision theory apparently not. I grant, of course, that it is very artificial to turn such cases into a measuring device for ranking functions.

Where to look for a better representation of rankings in belief change? Let us return to revisions and contractions as specified in definition 4. Given the underlying ranking function κ, they can clearly be iterated as well: $E{*}\langle A_1,...,A_{n+1}\rangle = E{*}\langle A_1, ...,A_n\rangle{*}A_{n+1}$, and likewise for contraction. Note that it did not matter how we fix the parameter $n > 0$ implicit in a single revision (which was defined via A,n-conditionalization). But, of course, the parameter makes a lot of difference for iterated revisions. For studying them we should thus fix the parameter to be the same in all revisions considered, preferably $n = 1$ throughout. However, this imports a cardinal arbitrariness which may compromise the investigation from the outset. By contrast, iterated contraction does not depend in this way on arbitrary decisions; definition 4 suffices to uniquely determine it relative to a given ranking function. This consideration leads me to pursue our question solely in terms of iterated contractions.

Let us observe, first, that iterated contractions are not reducible to a single contraction. The only tempting thought may be that $E \div \langle A,B \rangle$ is the same as $E \div \langle A \cup B \rangle$. But of course it is not. $E \div <A \cup B>$ must not contain the belief that $A \cup B$, whereas $E \div \langle A,B \rangle$ must only delete the beliefs in A and in B, but may retain the belief in $A \cup B$.

A bit less trivial is the observation that iterated contraction is even not reducible to (simultaneous) multiple contraction which has been inquired in belief revision theory by Fuhrmann and Hansson (1994). On p. 44 they mention that multiple and iterated contraction cannot be the same because the order of contractions may matter in an iteration – Hansson (1993, p. 648) has a nice example in which commutativity of iterated contractions intuitively fails –, but not in a multiple contraction in which all propositions have to be contracted at once. This is confirmed in ranking theory which also entails the non-commutativity of iterated contractions:

Theorem 2: $E \div \langle A,B \rangle \neq E \div \langle B,A \rangle$ iff $E \subseteq A,B$, $\kappa(B \mid \overline{A}) = 0$ or $\kappa(A \mid \overline{B}) = 0$, and $\kappa(\overline{B} \mid \overline{A}) < \kappa(\overline{B} \mid A)$ (which is equivalent to $\kappa(\overline{A} \mid \overline{B}) < \kappa(\overline{A} \mid B)$).

Constructing the proof shows precisely how the failure of commutativity comes about under these conditions. Roughly, the point is that A is positively relevant to B (and vice versa) and that the additional conditions thus have the effect either that $A\overline{B}$ is disbelieved (or „if A, then B" believed) after contracting first by A and then by B, but not after the reverse contraction, or that $\overline{A}B$ is disbelieved (or „if B, then A believed) after contracting first by B and then by A, but not after the reverse contraction (or that both is the case).

This observation raises two issues. First, there is the side question whether multiple contraction can also be explained in terms of ranking functions. Yes, it can. Suppose that $\{A_1,...,A_n\}$ are to be contracted from the core E of κ (all of which are assumed to be believed in E, in order to avoid triviality). Choice contraction of at least one of $A_1,...,A_n$ may then be defined as the single contraction of $A_1 \cap ... \cap A_n$, as Fuhrmann and Hansson (1994, p. 72) have observed. Package contraction of all of $A_1, ...,A_n$ may be defined in the following way: Let $B_1,...,B_m$ be those atoms of the

Boolean algebra generated by $\{A_1,...,A_n\}$ which are subsets of $\overline{A}_1\cup...\cup\overline{A}_n$. Clearly, all of $B_1,...,B_m$ are disbelieved in κ. Let C_1 be the union of the least disbelieved of these atoms, C_2 the union of the second least disbelieved, etc., and C_k the union of the most disbelieved. Now contract κ first by C_1, then by C_2, and so on until none of $A_1,...,A_n$ is believed any more (this procedure may stop before one has reached C_k). The core E' of the resulting ranking function κ' may then finally be defined as the package contraction $E \div \{A_1,...,A_n\}$. Thus one may prove that package contraction so defined satisfies all the postulates listed in Fuhrmann, Hansson (1994, pp. 50-55), i.e. the postulates (success), (inclusion), (vacuity), (relevance), (failure), and (uniformity) in the package version. These postulates do, however, not completely characterize package contraction as defined, because they generalize only (K÷1) - (K÷6); the appropriate generalizations of (K÷7) and (K÷8) are only conjectured in Fuhrmann, Hansson (1994, pp. 55-57).[9]

Mainly, however, theorem 2 suggests the question how a doxastic dependence or independence between A and B shows up in iterated contraction. The answer is straightforward: At least one of AB, $A\overline{B}$, $\overline{A}B$, or $\overline{A}\,\overline{B}$ must have rank 0. Suppose $\kappa(AB) = 0$ (if one of the other conjunctions has rank 0, the corresponding assertions hold). Then we have:

(PosRel) A is positively relevant to B w.r.t. κ, i.e. $\kappa(AB) + \kappa(\overline{A}\,\overline{B}) < \kappa(A\overline{B}) + \kappa(\overline{A}B)$ iff $E \div \langle A,B\rangle \subseteq \overline{A}\cup B$ or $E \div \langle A,B\rangle \subseteq A\cup\overline{B}$ or both, i.e. iff „if A, then B“ or „if B, then A“ or both are believed in $E \div \langle A,B\rangle$ (or, for that matter, in $E \div \langle B,A\rangle$).[10]

(NegRel) A is negatively relevant to B w.r.t. κ, i.e. $\kappa(AB) + \kappa(\overline{A}\,\overline{B}) > \kappa(A\overline{B}) + \kappa(\overline{A}B)$ iff E $\div \langle A,B\rangle \subseteq A\cup B$, i.e. iff „if non-$A$, then B“ is believed in $E \div \langle A,B\rangle$ (= $E \div \langle B,A\rangle$).[11]

(Irrel) A is irrelevant to B w.r.t. κ, i.e. $\kappa(AB) + \kappa(\overline{A}\,\overline{B}) = \kappa(A\overline{B}) + \kappa(\overline{A}B)$ iff none of $A\cup B$, $A\cup\overline{B}$, $\overline{A}\cup B$, and $\overline{A}\cup\overline{B}$ is a superset of $E \div \langle A,B\rangle$ (= $E \div \langle B,A\rangle$), i.e. iff none of the material implications between A or $\overline{A}$ and B or $\overline{B}$ is believed in $E \div \langle A,B\rangle$.

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[9] Sven Ove Hansson tells me that he does no longer believe in these conjectures.

[10] At least one of the material implications is believed in $E \div \langle A,B\rangle$ iff at least one of them is believed in $E \div \langle B,A\rangle$. Theorem 2 only says that there may be differences concerning which one(s) is(are) believed after the two contractions.

[11] The apparent asymmetry between (PosRel) and NegRel) comes about because our starting point $\kappa(AB) = 0$ entails that AB is not disbelieved and hence „if A, then non-B“ not believed in $E \div \langle A,B\rangle$.

What these conditions do, hence, is precisely to provide „operational" definitions of doxastic dependence and independence. And they do so, I find, in an intuitively appealing way: Suppose AB is not disbelieved. Thus neither A nor B is disbelieved, but possibly both are believed. Contraction by A then results in a doxastic state which is neutral on A, i.e. in which neither A nor $\overline{A}$ is believed. Likewise, contraction by B results in a state which is neutral on B. Hence, iterated contraction by A and by B (in either order) results in a state which is neutral on both, A and B. But possibly some material implications between A or $\overline{A}$ and B or $\overline{B}$ survive, and the kind of dependency between A and B should manifest itself precisely in which of these implications survive; in particular, independence between A and B should obtain precisely if none of them are maintained, as (Irrel) says.

We can immediately extend these conditions to „behavioral" definitions of conditional dependence and independence: Consider three propositions A, B, and C. At least one of the conditional ranks $\kappa(AB|C)$, $\kappa(A\overline{B}|C)$, $\kappa(\overline{A}B|C)$, and $\kappa(\overline{A}\,\overline{B}|C)$ must be 0. Assume without loss of generality that $\kappa(AB|C) = 0$. Then we have:

(CPosRel) A is positively relevant to B given C w.r.t. κ, i.e. $\kappa(AB|C) + \kappa(\overline{A}\,\overline{B}|C) < \kappa(A\overline{B}|C) + \kappa(\overline{A}B|C)$ iff either $\overline{A}\cup B\cup\overline{C}$ or $A\cup\overline{B}\cup\overline{C}$ (or both) is a superset of, i.e. believed in $E \div \langle \overline{C}, A\cup\overline{C}, B\cup\overline{C}\rangle$.

The corresponding conditions (CNegRel) and (CIrrel) hold for conditional negative relevance and conditional irrelevance.

These conditions look somewhat less perspicuous than the previous ones. The effect of the threefold appearance of $\overline{C}$ in the iterated contraction is to restrict all doxastic changes to (the possibilities in) C, and then the point is quite the same; (CIrrel) says, for instance, that A and B are independent given C iff none of the material implications „if C and A', then B'" ($A' \in \{A, \overline{A}\}$, $B' \in \{B, \overline{B}\}$) survives the three contractions.

The six conditions from (PosRel) to (CIrrel) are assertions about a ranking function κ. But as I suggested, they are also plausible assertions about (un-)conditional dependence and independence in an intuitive sense. This raises urgent questions: What are, intuitively, the properties of doxastic dependence and independence? Or, if that makes sense, what should they rationally be? Answers would have many consequences for iterated contractions (via these conditions) and for iterated revision (via the Levi Identity). Conversely, how are iterated contraction and revision intuitively or rationally to be expected to behave? Again, answers would determine a lot about doxastic dependence and independence. Either way, I have the discussion where I want to have it. Belief revision theory must think about doxastic dependence and independence and their properties; otherwise it is bound to be insufficient on its home field, revisions and contractions.

4. A Representation Result

Relative to ranking functions, anyway, doxastic dependence and independence are clearly defined and hence their „behavioral“ consequences uniquely determined. I do not want to engage now into an argument about the intuitive expectedness or acceptability of the properties of ranking dependence and independence. Generally, one can say that they are (almost) the same as those of probabilistic dependence and independence and agree with the graphical criterion of d-separation.[12] Hence, the question of intuitive acceptability is the same for both frameworks.

In this final section I rather want to address the question whether our observations open a way for a *complete* operational definition of ranking functions, i.e. to which extent ranking functions are conversely determined by suitable properties of iterated contraction via (Ord) and (PosRel) – (CIrrel). This seems to be a standard problem of measurement theory. Hence one should look for advice in the theory of difference measurement (cf., e.g. Krantz et al. 1971, ch. 4) or in the theory of probability measurement (cf. Domotor 1969, Krantz et al. 1971, ch. 5, or Fine 1973, ch. II) which proceeds from comparisons of unconditional and/or conditional probabilities and/or a qualitatively given independence relation.

However, it was not clear to me how to carry over these parts of measurement theory. One reason is that a complete axiomatization of conditional dependence or independence by itself is apparently still unknown (cf. Spohn 1994). Another reason is that, if measurement is based on probability comparisons, the usual route is to identify so-called standard, i.e. equally distanced sequences; however, ranking function would lose their point, if they had to embody standard sequences.

Still, it does not seem so difficult to achieve a representation of ranking functions; after all, things are vastly simplified by the fact that only natural numbers are possible measurement results. So let us start from some ordering $\leq$ of disbelief

[12] „Almost“ refers to the fact that only some very far-fetched differences have been discovered. Cf. Spohn (1994).

which is reflected in contraction via (Ord) and from some conditional non-negative relevance relation, denoted as $A\Delta B|C$, which is reflected in iterated contraction via the disjunction of (CPosRel) and (CIrrel). The goal then is to thereby represent a ranking function κ such that for all A,B,C: $A\leq B$ iff $\kappa(A) \leq \kappa(B)$ and $A\Delta B|C$ iff $\kappa(AB|C) + \kappa(\overline{A}\,\overline{B}|C) \leq \kappa(A\overline{B}|C) + \kappa(\overline{A}B|C)$. I am content with presenting one inelegant way to achieve this goal; it is certainly open to improvement.

First, the necessary properties of $\leq$ are obvious (where $A\approx B$ is defined as $A\leq B$ and $B\leq A$, and $A<B$ as $A\leq B$, but not $B\leq A$):

(1) $\leq$ is a weak order, i.e. transitive and connected,
(2) if $A\leq B$, then $A\approx A\cup B$.

This entails that there is a sequence of non-empty propositions $E_0,\ldots,E_n$ which partition W such that $E_0<\ldots<E_n$ and $A\approx E_j$ iff $A\cap E_j \neq \emptyset$ and $A\cap E_i = \emptyset$ for all $i < j$ (and hence $A\approx E_j$ for all non-empty $A\subseteq E_j$). Thus, $\leq$ is completely captured by the sequence $E_0,\ldots,E_n$, and we need only determine the ranks of $E_0,\ldots,E_n$.

Here, conditional non-negative relevance may help in the following way. First, we can reduce any such relevance to one among propositions constructed from the sequence $E_0,\ldots,E_n$: Let A,B,C be any three propositions. Then there must be $i,j, k,l\leq n$ such $ABC\approx E_i$, $A\overline{B}C\approx E_j$, $\overline{A}BC\approx E_k$, and $\overline{A}\,\overline{B}C\approx E_l$. We may assume without loss of generality that $E_i\leq E_j,E_k,E_l$. If $E_l<E_j$ or $E_l<E_k$ should be the case, then A is obviously positively relevant to B given C. This is stated in a necessary condition:

(3a) if $E_l < E_j$ or $E_l < E_k$, then $A\Delta B|C$.

Let us consider then the other case where $E_j,E_k \leq E_l$. Again, without loss of generality we may assume that $E_i \leq E_j \leq E_k \leq E_l$. Now define:

$$F_i = E_i\cup\ldots\cup E_n,$$
$$F_{ik} = E_i\cup\ldots\cup E_{k-1},$$
$$F_{ijkl} = E_i\cup\ldots\cup E_{j-1}\cup E_k\cup\ldots\cup E_{l-1}.$$

Then, we obviously have $F_iF_{ik}F_{ijkl} \approx E_i$, $F_iF_{ik}\overline{F}_{ijkl} \approx E_j$, $F_i\overline{F}_{ik}F_{ijkl} \approx E_k$, and $F_i\overline{F}_{ik}\overline{F}_{ijkl} \approx E_l$. This entails the further necessary condition:

(3b) if $E_i \leq E_j \leq E_k \leq E_l$, then $A\Delta B|C$ iff $F_{ik}\,\Delta\, F_{ijkl}\,|\, F_i$.

Hence, we can confine ourselves to considering only such triples of F-propositions. They can be used to measure the rank distances between members of the sequence $E_0,...E_n$ in the following way: Let us say that the distance beween E_i and E_{i+1} is minimal iff for all $k > i$ $\overline{F}_{ik} \,\Delta\, F_{ii+1kk+1} \,|\, F_i$ and for all $k > i$ $F_{ki} \,\Delta\, F_{kk+1ii+1} \,|\, F_k$. If one observes first the intended representation of non-negative relevance and second the fact that $\overline{A}\Delta B|C$ expresses the conditional non-positive relevance of A to B, then it becomes clear that this definition captures the intuitive meaning.

If the distances between E_i and E_{i+1} were minimal for all i = 0,...,n-1, then E_0, ...,E_n would indeed form a standard sequence and we were finished. However, to require so much minimality would be uninterestingly restrictive. The following structural condition is much weaker (for which we define $A \perp B|C$, i.e. A is irrelevant to B given C, iff $A\Delta B|C$ and $\overline{A}\Delta B|C$):

(S) Whenever the distance between E_i and E_{i+1} is not minimal, then there are k and $l \geq k$+2 such that $F_{ik} \perp F_{ii+1kl} \,|\, F_i$, if $k > i$, and $F_{ki} \perp F_{klii+1} \,|\, F_k$, if $k < i$.

Numerically, (S) has the effect that each non-minimal distance is the sum of smaller distances and thus in the end a unique multiple of the minimal distance. Clearly, (S) is a non-necessary condition; relations ≤ and Δ induced by some ranking function may or may not behave according to (S). But (S) is sufficient for representing ranking functions:

As I said, (S) uniquely fixes somes function f defined for $\langle i,j\rangle$ with $i < j \leq n$ such that $f(i,i+1) = 1$ iff the distance between E_i and E_{i+1} is minimal and such that $f(i,j)$ measures the distance between E_i and E_j in multiples of the minimal distance. However, it does so only if non-negative relevance is well-behaved. So, given (S), the following condition is also necessary:

(4) $F_{ik} \,\Delta\, F_{ijkl} \,|\, F_i$ iff $f(i,j) \geq f(k,l)$.

So, all this sums up to the following

Theorem 3: Whenever the relations ≤ and Δ satisfy conditions (1) - (4) and (S), then there is a ranking function κ such that for all A,B,C: $A \leq B$ iff $\kappa(A) \leq \kappa(B)$ and $A\Delta B|C$ iff $\kappa(AB|C) + \kappa(\overline{A}\,\overline{B}|C) \leq \kappa(A\overline{B}|C) + \kappa(\overline{A}B|C)$. There is only one such ranking function κ' such that $\kappa'(E_{i+1}) - \kappa'(E_i) = 1$ if the distance between E_i and E_{i+1}

is minimal. If κ'' is another ranking function thus represented, then there is some natural number $\alpha > 0$ with $\kappa'' = \alpha\kappa'$.

Hence, ranking functions are measured on ratio scales; more uniqueness could obviously not be expected.

Of course, not each ranking function may be uniquely represented in this way. For instance, if $n = 2$, i.e. if κ distinguishes only three ranks $\kappa(E_0) = 0$, $\kappa(E_1) = x$, and $\kappa\,(E_2) = y$, then the above machinery helps to determine whether $2x = y$ or $2x < y$ or $2x > y$; but if one of the latter two holds, there is nothing to further determine x. Still, theorem 3 can and should be improved. Structural conditions weaker then (S) may do as well, and the necessary conditions (3) and (4) can and should be expressed in a nicer way.

But formal optimality was not my aim here. The point of the exercise was only to show that there are conditions sufficient for representation, namely the conditions (1) - (4) and (S), which are expressible in terms of ≤ and Δ[13] and thus, via (Ord), (CPosRel), and (CIrrel), ultimately in terms of iterated contractions. Hence, iterated contractions alone suffice for fixing the structure of ranking functions.[14]

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[13] This is obvious for the conditions (1) - (3) and (S). But it holds for (4) as well since any distance $f(i,j)$ can be built up from smaller distances in only finitely many ways; hence, such clauses as $f(i,j) = n$ and $f(i,j) \geq f(k,l)$ can be expressed in a finite condition on ≤ and Δ.

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