# Shapiro's and Hellman's Structuralism

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**Abstract**

I give brief descriptions of and motivations for Stewart Shapiro's and Geoffrey Hellman's positions in mathematical structuralism and criticize each position in turn. Shapiro explains mathematical objects as 'places in structures', conceived as akin to offices, occupiable by arbitrary things and individuated by their interrelations. I argue that while places as occupiables are much like the 'places' of relations and thus not too worrisome, the two aspects of places – occupiability and what I call 'relational essence' – don't go well together. Therefore Shapiro's structuralism doesn't constitute a substantial advance over traditional platonism. Furthermore, mathematical practice indicates that mathematical objects aren't needed anyway to account for the information content of mathematics. Accordingly, in Hellman's version of structuralism mathematical propositions do not refer to mathematical objects; rather, they are implicit generalizations over logically possible systems. Statements P of a mathematical theory have to be analyzed as "Necessarily, for all systems of type so-and-so, P", where the theory also says that it is logically possible that there be systems of type so-and-so. I claim, however, that the logical possibility of structural conditions isn't required for mathematical reasoning about them. Thus Hellman's structuralism isn't adequate as an account of mathematics. Finally, I hint at my own half-baked views on what mathematics is about.

## 1 Introduction

Two versions of structuralism

I am a structuralist. I find myself situated somewhere between Stewart Shapiro and Geoffrey Hellman – which is a nice place to be, I guess.

When I first got acquainted with structuralism I sympathized with Shapiro's (1997) account, because it seemed to be a plausible way of 'obtaining' mathematical objects, namely, as *places in structures*, where structures are conceived as universals. However, after I had taken a much closer look at the details of his account of structures and places, I became dissatisfied with it. I now think that, while places are not terribly problematic objects, they still aren't the *mathematical* objects. After sympathizing with realism about mathematical objects, I now tend to think that there are no such things.

Mathematical objects or no, I have for a long time thought, and still believe, that mathematical theories, even theories that seem to be about one specific intended model, are in some sense about *all* systems of objects having the right structure. This brings me into the neighborhood of Geoffrey Hellman (1989), who has developed

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a detailed account along these lines, an account that is nominalist with respect to mathematical objects and is based on a primitive notion of logical possibility.

A lot of what Hellman says in his publications sounds very reasonable to me. But in the end, Hellman's account doesn't seem quite right to me either. So I will have to point out what I don't like about Hellman's account, and then give you my very rough ideas of what one should say instead.

So, I will briefly describe Shapiro's version of structuralism, especially his notion of places in structures. I will then tell you why I think that places can't play the metaphysical role Shapiro wants them to play. Then I will say some words on structuralism *without* mathematical objects, which brings me to Hellman. I will argue that his account, while it certainly is in the right spirit, doesn't yield an adequate representation of what goes on in mathematics. Finally, I will point in the direction I think structuralism should take – and thus (who knows) maybe start my *own* trend in the philosophy of mathematics.

## 2 Shapiro's structuralism

I start with Stewart Shapiro's account, redescribing it in my own way, the way in which I think it becomes most plausible.

### 2.1 Systems and structures

The natural-number structure: a property of systems

Consider the natural numbers: $0, 1, 2, \ldots$ It is unclear whether they exist, and if so, what they are, what their nature is. But it *is* clear what *structure* the system of the numbers is supposed to have: the number system consists of a collection of objects (called "natural numbers") and certain relations on that collection, say, a successor relation; and this successor relation interrelates the numbers in such a way that the Peano axioms are satisfied.

This way of being interrelated by a two-place relation is a (second-order) property that systems can have, a one-over-many. We can also describe it as a second-order relation between (1) a collection of objects and (2) a *relation* on that collection – although I suppose Shapiro would not put it that way. This property of systems, this second-order relation, which is expressed by the Peano axioms, is the *natural-number structure*.

Systems having the natural-number structure

The natural-number *system* $(\mathbb{N}, \mathrm{Succ})$ – if it exists – *has* that structure. So do, *inter alia*, certain set-theoretical systems, for example,

- the *Zermelo numerals*: $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \left\{\{\{\emptyset\}\}\right\}, \ldots$ (where "$y$ is the successor of $x$" means "$y = \{x\}$"),
- or the *finite von Neumann ordinals*: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\left\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\right\}$, $\ldots$ (where "$y$ is the successor of $x$" means "$y = x \cup \{x\}$")

– assuming that *sets* exist.

Dubious exemplars, well-understood property

This property of systems, the natural-number structure, seems to me unproblematic: We may disagree about whether there *are* any systems which have that structure – especially systems consisting of physical objects. And if systems with that structure exist, we may disagree about whether there is one special, distinguished system among them that contains *the* numbers, instead of objects like sets, which can be used as numbers, which can play the role of numbers.

We may disagree about systems which *have* the natural-number structure. But the natural-number structure itself, that property of systems – I think we know reasonably well what we are talking about when we talk about *that* structure.

So the natural-number structure, considered as a property of systems, is something I believe in. It is real enough for me to take it seriously – whatever that means precisely. Similarly for other structures: the complex-number structure, the Euclidean-plane structure, and so on.

## 2.2 Places in structures

Places in structures

Now that we have understood (to some degree) what a structure is, we can go on to introduce places in structures.

Consider the Zermelo numerals. Every object in this system plays a particular role. For example, the set $\{\{\emptyset\}\}$ is the successor of the successor of the only nonsuccessor in the system. That's what is special about it, in the context of the Zermelo numerals. It's the same thing with the set $\{\emptyset, \{\emptyset\}\}$ in the context of the finite von Neumann ordinals: it is the successor of the successor of the only nonsuccessor – but with another successor relation and another domain. The two sets play the same *role* in their respective systems, the role that the number 2 plays in the natural-number system.

They play the same role, but we might also say they are in the same *place* in their respective systems, the place that 2 occupies in the natural-number system. It's no wonder we call this place the *2-place* in the natural-number structure.

Objects of systems occupy places in a structure

So, for every system having the natural-number structure, its *zero*-object occupies the zero-*place* in that system. Analogously, each zero-object's *successor* – say, the set $\{\emptyset\}$ or the number 1 – occupies the 1-place in the given system. And so on.

Thus the natural-number structure has countably many places: the zero-place, the 1-place, the 2-place, and so on. These are the places in the natural-number structure. All the zero-objects occupy the *same* place in the natural-number structure, only in different systems. All the 1-objects occupy the same *distinct* place in the natural-number structure, in their respective systems.

Indiscernibility of places

Now consider the Euclidean plane instead of the natural numbers. Since all the points in the plane have the same geometric properties, every point plays the same role as every other. But if we want places to be mathematical objects, then there must be for each point a separate place. If these places are the mathematical objects we want, if they *are* the points in the Euclidean plane, then it seems they are all *indiscernible*, though distinct.

There has been a lot of discussion about whether indiscernibility of places constitutes a serious problem for Shapiro's *ante rem* structuralism. But this phenomenon doesn't worry me: it seems to me that places in structures are much like the 'places' of relations.

For example, the relation "$x < y$" has two places: the $x$-place and the $y$-place. You input a number into each of the places, and the $<$-relation then tells you whether the $x$-object is less than the $y$-object or not.

Structures as $\nu$-place relations

If you consider the natural-number structure as a property of whole systems, then it is a one-place relation: you input a system, and it either has the natural-number structure or it hasn't. If you consider the natural-number structure as a relation between a *collection of objects*, on the one hand, and a *relation* on that collection, on the other, then it is a two-place relation: you input a set or collection into the one argument place, and a relation into the other.

If, however, you consider the natural-number structure as a relation between countably many *single objects* and one relation, then it is an $(\omega+1)$-place relation: you input one object into the zero-place, one object into the 1-place, and so on, countably many times, and finally you input a relation into the structure's successor-relation place. And then these objects and this relation either have the right kind of second-order relation or they haven't.

Of course you can't do it this way in practice. In practice, you specify *implicitly* what goes where, by specifying the domain as a whole, and its successor relation, via general characterizations. But I think the places of the natural-number structure can be understood by pretending that we can input an infinity of arguments into a relation with an infinity of argument places.

Argument places of symmetric relations

And so what if these argument places sometimes are indiscernible? Look at the argument places of the relation "$x$ is the same age as $y$" (or of any other symmetric relation): those two places aren't discernible either.[1] So, symmetric relations somehow manage to have two distinct, but indiscernible, argument places. Then it is not such a big step to accepting that in some sense the Euclidean-plane structure has infinitely many indiscernible argument places.

I do not want to maintain that we have a good understanding of what places in this sense are. We do not. But the task of better understanding argument places is one we are burdened with anyway, independently of mathematics. And we do have a *slight* understanding of places: we know what one can do with them, namely, input things in them in order to obtain some output. So, even though the notion of places is not well understood, I have no scruples about accepting it into an account of structures.

## 2.3 Are the office places the mathematical objects?

Are the places the mathematical objects?

It is the *next* step that I do not accept: to declare that these places are the mathematical objects.

What I have said up to now did not necessarily pertain to mathematical objects. It was about places in structures, conceived as somewhat like offices to be occupied, or like roles to be played, by other objects. Shapiro says this is one *perspective* on places in structures, namely, the *places-are-offices* perspective. I suggest that we should rather consider this as one *concept* of places. I call places in this sense the *office places* of a structure.

What are mathematical objects like?

Are the office places of mathematical structures the corresponding mathematical objects? Well, what are mathematical objects like (if they exist)? What are numbers like?

I suppose virtually all of you have read Paul Benacerraf's paper "What Numbers Could Not Be" (1965). Not everybody draws Benacerraf's own conclusion: that numbers aren't objects at all (which presumably means that they don't exist). But it seems that most readers conclude that *if* the numbers exist then their mathematical properties consist in their *arithmetical relations* to each other: the number 2 is the successor of 1, the predecessor of 3, the first prime, etc. – otherwise it wouldn't *be* 2. And furthermore, the mathematical properties of the numbers do *not* include, e.g., any *set-theoretical* relations between them: whether 2 is an element of 4 is just a bad

[1] Of course, if you look at a linguistic expression or other representation of that relation you have discernible *place-holders* for argument designations. But that is all on the level of language, while the relation itself, and its places, are *not* linguistic phenomena.

question, it doesn't make much sense.[2]

We might abbreviate this by saying that the numbers – or mathematical objects generally – have a *relational essence*. Here, the negative part of the Benacerraf moral is meant to be included as well as the positive part. That is, it is essential to the numbers that they can be added to and multiplied with each other, and stand in the successor relation; *and* it is essential to the numbers that they are *not* elements of one another (or open or closed, or parallel to each other, etc.), that this notion doesn't even make good sense. That's what I mean when I say, numbers or other mathematical objects have a relational essence.

Essence places
(= mathematical objects)

For Shapiro, this is another perspective on places in structures: when we say "$2+2$ is 4" or "5 is prime", we conceive numbers – the places in the natural-number structure – not as 'offices' to be occupied by objects from some background ontology, but rather as objects in their own right; we take the *places-are-objects* perspective, as Shapiro calls it. I suggest that we should think of this not as another *perspective* on places, but rather as a different way of understanding the word "place", as a different *concept* of places in structures. I call places in this sense *relational-essence places*, or "essence places" for short.

So, the way *I* tell Shapiro's story – which is not the way he tells it himself – there are two concepts of places in structures: on the one hand, office places, which are like offices or slots that can be occupied by objects, or like roles to be played; on the other hand, essence places, which stand in certain mathematical relations to each other (relations which are peculiar to the structure they belong to), and which stand *only* in *those* mathematical relations. Essence places certainly are mathematical objects; and mathematical objects are the essence places of their respective structures – if they exist.

What makes Shapiro's account attractive in my eyes is that he doesn't content himself with merely pulling mathematical objects – that is, essence places – out of his hat, by postulating them, as traditional platonists do. Rather, he tries to present plausible candidates for the mathematical objects, namely, the office places of structures.

Office places
= essence places?

But are office places and essence places really the same? I don't think so.

Shapiro doesn't distinguish between office places and essence places. Mostly he just *presupposes* that what I call office places and essence places are the same. But there are a few occasions where he tries to motivate this identification:

> In contrast to this office orientation, there are contexts in which the places of a given structure are treated as objects in their own right, at least grammatically. That is, sometimes items that denote places are bona fide singular terms. *We say that the vice president is president of the Senate, that the chess bishop moves on a diagonal, . . .*
>
> My perspective . . . presupposes that statements in the places-are-objects perspective are to be taken literally, at face value. Bona fide singular terms, like "vice president," "shortstop," and "2," denote bona fide objects. (Shapiro 1997, p. 83; my italics)

Offices treated as objects
– allegedly

What do these passages tell us? When someone says, "The chess bishop moves on a diagonal", and they are not talking about a particular *concrete* chess piece, then "the chess bishop" denotes a certain *role* concrete chess pieces can play. Analogously, when someone says, "5 is prime", and they are not talking about the 5-object of

[2] It does make more sense than the question "Is 2 an element element?" For example, "If $2 \in 4$ then $2 \in 4 \cup 7$" seems somewhat meaningful and even plausible to me.

some particular 'concrete' system having the natural-number structure, like, e. g., the Zermelo numerals, then "5" would denote an *office place* in the natural-number structure. If that is the correct understanding of "5 is prime", then this office place would indeed have the primeness property, i. e., an arithmetical property that is characteristic of certain *numbers*, certain *relational-essence* places. Thus office places would indeed be essence places.

Hidden generalizations about office occupants

But is it really the *role* of bishop in chess that moves on a diagonal? Or isn't it rather concrete chess pieces which move; and talking about 'the chess bishop' is just a convenient way of talking about whatever plays that role in a given game? And is it really the *offices* of vice president and president of the Senate which coincide? Maybe they are, but we can well imagine a changed constitution where the occupant of the one office *doesn't* automatically occupy the other as well.

Analogously we might ask, is the 5-*office* really prime? What does "prime" even mean for office places of the natural-number structure? Truths like "5 is prime" do not give us reason to think that it is the 5–*office place* which is prime.

Offices treated as objects – really

By contrast, think about the sentence

> "The office of president of the USA *has a special seal*."

This sentence really does refer to the office as an object in its own right. It is not a disguised way of talking about arbitrary *occupants* of the office. – Or we might perhaps say,

> "The 6-place of the natural-number structure *is just as occupiable as* the zero-place."

These sentences do treat offices as objects in their own right. But the properties they predicate of them are ones fit only for offices, not for their occupants.

I think Shapiro's preferred face-value reading of his example sentences does not work. He must maintain that in these sentences the properties expressed are predicated of offices treated as objects in their own right, independently of any occupants. But the properties or relations expressed in Shapiro's everyday sentences,

> "The vice president is president of the Senate",
>
> "The chess bishop moves on a diagonal",

do not make sense for roles or offices; they only make sense for their occupants. So these are disguised generalizations.

"5 is prime": '*numbers*-are-objects' or general

By contrast, *mathematical* sentences can be read in two ways:

- Either "5 is prime" talks about a number, about an essence place, and it predicates of it a property that is fit for mathematical objects, but has no obvious connection to office places.
- Or "5 is prime" is really a generalization about arbitrary *occupants* of the 5-place, and then it doesn't predicate a particular property of the place ('*the* property of being prime'), but rather states that the occupants have *their system's* primeness property, whatever that may be.

It is just not the case that *the* property of being prime – a property distinguishing certain numbers – is said to distinguish certain of the natural-number structure's office places.

So there is still no reason to believe that office places are essence places, that office places are the mathematical objects. Thus, contrary to appearances, Shapiro does not give us suitable candidates for the mathematical objects.

*Identifying* office places and essence places?

But we are philosophers; can't we just *postulate* that office places are essence places? – I don't think so. Not only do we not have evidence that office places *are* essence places; there is in fact evidence that office places are *not* essence places:

Office places are not essence places

On the *essence* places, say, on the numbers, there are certain special relations, e. g., *the* successor relation. You can permute the numbers in ways such that the resulting system with its new successor relation again has the natural-number structure. But this new successor relation is not *the* successor relation; it is merely one of many *ersatz* relations.

By contrast, on the *office* places of the natural-number structure, there is no such distinguished, privileged successor relation. Of course, if you want to define a successor relation on the number office places, there is one relation that is particularly easy and natural to specify:

> Office place q is an *office-place successor* of office place p iff in any system $S$ with the natural-number structure, q's occupant is the $S$-successor of p's occupant.

Thus you get: 0-place $\curvearrowright$ 1-place $\curvearrowright$ 2-place $\curvearrowright$ 3-place $\curvearrowright \ldots$ But this relation is in no way metaphysically special; it is only *pragmatically* special. You could just as well define successorship among office places such that you get the ordering 1-place $\curvearrowright$ 0-place $\curvearrowright$ 3-place $\curvearrowright$ 2-place $\curvearrowright \ldots$, or in lots of other ways, still obtaining systems of office places that have the natural-number structure.

So, the 'natural' or 'canonical' successor relation on the number office places is not *the* successor relation on those places. '*The* successor relation on the number office places' just doesn't exist. Therefore the number office places do not have what it takes to be the numbers, i. e., the number essence places.

Essence places are not office places

Symmetrically, the numbers do not have what it takes to be the number office places: What is special about office places is that they are (in some sense) *occupiable* by arbitrary objects. In what sense can the *numbers* be 'occupied' by objects? The way for a system's objects to 'occupy' the numbers, the essence places of the natural-number structure, consists in there being an isomorphism from that system to the system of the numbers. 'Occupying' the number zero, the zero–essence place, means being *mapped onto* zero by that isomorphism; 'occupying' the number 1 means being mapped onto 1. And so on.

But if that were all it takes to be an office place in the natural-number structure, then any old system having the number structure could do the job just as well. Because if a system has the natural-number structure then there is automatically an isomorphism to any other system with that structure (because second-order arithmetic is categorical). For example, the Zermelo numerals or the finite von Neumann ordinals have an equal right to be considered as the office places of the natural-number structure as do the numbers themselves.

So the numbers are not suited any better to be the natural-number structure's office places than are the objects of arbitrary other systems having that structure.

Mathematical objects: by stipulation only

Therefore I conclude that the numbers, or in general the essence places of a structure, are not the office places of that structure. But then office places do not succeed in constituting plausible essence places, plausible mathematical objects. And thus Shapiro's structuralism leaves mathematical objects as mysterious as before:

if we want to believe in them we have to stipulate them out of thin air, just like in traditional platonism.

## 2.4 Mathematical objects are unnecessary

But I wonder: do we really need mathematical objects?

Are mathematical objects needed for semantics?

Paul Benacerraf, in "Mathematical Truth" (1973), says that we would like to have a Tarski semantics for mathematical discourse, and for that we need mathematical objects as denotations of singular terms like "2". But Tarski semantics, while technically clear-cut, is just a mathematical *model* of semantics, and not very satisfying really as an account of semantics. It leaves too many important questions open, e. g., where those referential relations come from.

Furthermore it seems to me that realism in ontology, coupled with Tarski semantics, does not give a very faithful account of how mathematical discourse acquires its content. The following argument is the one argument that is intuitively the most compelling for me in rejecting realism about mathematical objects:

The numbers . . .

Look at how we go about finding things out about the natural numbers. We talk as if we had one particular system of objects in our hands: the numbers. We talk about 8 and 17, and about adding and multiplying numbers, and so on. And we behave like we just know certain things about these objects and functions, e. g., the Peano axioms. And there are certain questions we just don't ask, like "What color does 5 have?", or "Does 5 have any elements?" Only a misguided philosopher would ask questions like that.

. . . vs. a group's elements

Now compare how we go about finding things out about, for example, all groups: We say, "*Let* G *be a group*, let $\circ$ be its multiplication, and let $e$ be its unit element", and then we go on talking as if we had one particular system of objects in our hands: the group G. We behave like we just know certain things about this system, e. g., the group axioms. And certain questions we just don't ask, like "What color is the unit element?", or "Do any of the objects of G have elements?", or "How many objects are there in G?"

But in this case nobody would conclude that after the preparatory formula "Let G be a group" we are indeed talking about one particular system: 'the arbitrary group' perhaps, whose objects do not have any mathematical properties besides those which follow from the group axioms, whose objects do indeed not even have a particular number! (It would have to be 'the arbitrary cardinality', I guess.) No, of course we are not talking about one particular group: we are proving theorems about whatever group someone might some day stumble upon.

No relevant difference in practice

Now, it seems to me that there is no relevant difference between these two cases, arithmetic and group theory. Yes, the axioms are different, and the one theory is categorical while the other is not. But I see no grounds for believing that in arithmetic we somehow make referential and epistemic contact with one particular system, while in group theory we do not. What is happening semantically and epistemically in these two cases is strictly analogous.

So, it looks like we can do the mathematics we do whether there are mathematical objects or not, because we don't treat theories that look like they are about *particular* systems any differently from theories that are about *arbitrary* systems of a certain structure type.

# 3 Hellman's structuralism

Whether the numbers exist or not doesn't make any difference to the way mathematics is practiced – except that as philosophers we have to tell a different story about the semantics of mathematical propositions. If there are no numbers, if mathematics is actually general where it seems to be about particular objects, then "2+2 is 4" does not refer to numbers 2 and 4. What is its meaning then? One structuralist answer comes from Geoffrey Hellman: modal-structuralism.

Another semantics of mathematical discourse?

## 3.1 Generality and triviality

I start with an intermediate step on the way to Hellman's position. If arithmetic isn't about 'the numbers' but rather about arbitrary systems having the natural-number structure, then the obvious thing to do would be to say, what "$2+2=4$" really means is

"∀": the danger of triviality

$$\forall \text{ systems S with the } \mathbb{N}\text{-structure:} \quad 2_S +_S 2_S \;=\; 4_S.$$

And mathematicians might well rest content with this, but then along comes a philosopher and says: "What if there are no systems that have the ℕ-structure? Then '∀ ℕ-systems S: $2_S +_S 2_S = 4_S$' would be *trivially* true, just like '∀ ℕ-systems S: $2_S +_S 2_S = 5_S$'!"

So, to take care of this worry, we amend our explication and say, arithmetic isn't just about all ℕ-systems which *actually* exist, but rather about whichever ℕ-systems *might possibly* exist. So, what "$2+2=4$" really means is

"□∀": triviality vanquished (?)

$$\Box\forall \text{ natural-number systems S:} \;\; 2_S +_S 2_S \;=\; 4_S,$$

"whichever way the world might logically be, $2+2$ is 4 in all natural-number systems S". And since it seems logically possible that there be infinitely many objects –

$$\Diamond\,\exists S\text{: S is a natural-number system}$$

– this explication of "$2+2=4$" is not in danger of being trivial.

(You may, however, get an analogous problem when you try to apply this method to set theory, at least if you want your systems to consist of physical objects: is it really logically possible that there be as many physical objects as there are sets? So, propositions of the type "□∀ set-theoretic-hierarchy systems S: ..." may again be in danger of all being trivially true.)

I have to add that this was not a faithful rendering of what Hellman says. I have left out lots of detail and have given you just enough of an idea of what he says to enable you to understand the criticisms that will follow.

## 3.2 Possibility is unnecessary

So, what is it I don't like about Hellman's proposal? A first clue is what Hellman calls the *categorical component* of each theory. His formalization of arithmetic starts with the assertion that systems with the natural-number structure are logically possible:

"◊∃": asserted or presupposed?

$$\Diamond\,\exists S\text{: S is an } \mathbb{N}\text{-system.}$$

In this way, the *hypothetical component* of arithmetic – the theorems "□∀ ℕ-systems S: ..." – is 'grounded', is saved from triviality.

But mathematicians usually don't assert things like that ("$\Diamond\exists\ldots$") in their everyday work. Only when they wax philosophical do they assert things like, "Of course the numbers exist!" In their everyday practice they just tacitly *presuppose* that the systems they seem to talk about exist, or that the structures they investigate are indeed instantiatable.

This doesn't prove much. Maybe mathematicians' shared conviction that their systems exist is so strong that they just don't need to state explicitly their belief in those systems. Maybe there is just no need to mention it, because everybody agrees.

But I don't think that's quite it. I think it is indeed just a presupposition, a working hypothesis, that the numbers (or the sets, or anyway systems with the required structure) exist (or might exist). Someone who believes this hypothesis false shouldn't bother doing arithmetic, at least not for its own sake, because it would be futile.

Of course it *is* important to some degree that the structures mathematicians investigate are 'consistent', logically possible, instantiatable, at least by systems of abstract objects. But maybe it is not always important; and maybe sometimes it doesn't matter at all.

Three kinds of premisses for proofs

Let me try to make this plausible. Consider three kinds of proof that mathematicians produce:

(a) Sometimes they start with something like "Let $\mathbb{N}$ be the system of the natural numbers", or maybe "Let S be a system containing nine objects and fulfilling structural requirements so-and-so", where they are *certain* that systems like these (might) exist. They proceed to develop logical consequences of the structural properties they have assumed, and then pronounce a categorical assertion: "In systems like that, such-and-such is the case."

(b) In another kind of proof they may be *uncertain* whether their starting hypothesis is true. For example, a mathematician might say, "Suppose Takahashi's Conjecture is true", and then she derives logical consequences of that assumption. In the end, she obtains an implication: "If Takahashi's Conjecture is true then such-and-such holds."

(c) In the third kind of proof it is supposed, or maybe even clear, from the outset that the hypothesis the mathematician starts with is *false*, logically false. You do this whenever you lead a proof by *reductio ad absurdum*. For example, you say, "Suppose there were a greatest prime number." Or you do it in a proof of an implication "If P then Q", when you say, "Suppose P is true and Q is false." You assume this even though you are certain that it cannot possibly be true.

Different kinds of content?

Now, the thing is this: In all these three kinds of proof, the same thing goes on, cognitively or epistemically. You assume that matters are a certain way – never mind whether that way for things to be is logically possible – and then you derive consequences. Assuming in this manner the *impossible* works just the same way as assuming the possible or starting from a virtual certainty.

But if Hellman is right then different things are going on in these three cases. In case (a), where you talk about a kind of system you believe in, you start by asserting, "There might be systems like this!" I'm not sure how Hellman would describe the second and third case, where you start with something uncertain or even impossible. Maybe that's mere symbol manipulation, to be interpreted formalistically? Anyway, here the 'categorical part' seems to play no role; you don't need the assertion that Takahashi's Conjecture is a logical possibility, or that there really *might* be a greatest prime number.

Modal-structuralism is inadequate w. r. t. practice

But these three cases are all analogous with respect to mathematical practice. For me, this implies that the sorts of content these kinds of proof deliver are the same. Since Hellman's characterization treats them completely differently, it must be inadequate. In a way, mathematics is about *more* than what is logically possible (e. g., "If $2+2 = 5$ then $(2+2)^3 = 5^3$").

Of course, if we have to choose among the available formalisms for characterizing mathematics, we may not find anything better. So either we have to invent a new kind of formalism, or we have to give up on formalizing our philosophical interpretation of mathematics.

## 4 Von Bülow's structuralism?

Maths as the science of conceivable structure

What do *I* think? What's my constructive counterproposal?

I haven't got more than very rudimentary and vague suggestions yet: I am quite certain that mathematics is nowhere about *particular* systems, like 'the numbers' or 'the Euclidean plane'. You might say it is about particular *structures* or types of structure, about particular ways systems might conceivably be (where not everything that's 'conceivable' is also logically possible).

Maths, the universe and everything

Mathematics is so general that there is no circumscribed realm of systems or things it is about.[3] (That's a point Hellman often makes, rightly, in different words.) There is not even a however-vaguely-circumscribed *spectrum* of realms of systems mathematics is about, e. g., the spectrum of the logically possible, as in Hellman's account. As soon as you circumscribe such a 'realm' or 'spectrum of realms', sooner or later mathematics will transcend it. I think this also happens to Hellman's account: his prohibition of quantifying over all possible systems, or of functions and relations between different possible 'universes', will not forever stop mathematicians.

Mathematical content

What is mathematics about then, if not about any 'circumscribed realm'? I like to think about mathematical theorems not as *propositions* with a well-defined, exhaustible content, but rather as *tools* with no pre-determined domain of applicability. They are applicable to *whatever* someone might specify some day.

Maybe I should better say that mathematics is not about arbitrary conceivable *systems* but about arbitrary *descriptions* or *representations* of (structural) kinds of systems. Some of these descriptions are of impossible systems, but as long as they have at least some halfway coherent parts you can still reason mathematically about those parts.

How to philosophize about mathematics

How to make this precise? How to formalize it? – I have doubts about formalizing a philosophy of mathematics. The formalism may be a nice model, but in the end it is still just more mathematics; and I don't believe you can fully understand mathematics just by doing more mathematics.

What then? The way I would go about trying to understand mathematics is by looking at how physical agents – machines with goals – can arrive at, and apply, mathematical knowledge (or mathematical information, or capabilities), whether primitive, like rudimentary counting, or sophisticated.

But that's all I can say for now.

[3]Though probably the real problem lies in attempting to circumscribe, or in supposing as circumscribed, 'everything there is' – or in believing in an absolutely general concept of existence.

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