What Is a Place in a Structure?*

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Brian: Look. There was this man, and he had two servants.
Arthur: What were they called?
Brian: What?
Arthur: What were their names?
Brian: I don't know. And he gave them some talents.
Eddie: You don't know?!
Brian: Well, it doesn't matter!
Arthur: He doesn't know what they were called!
Brian: Oh, they were called "Simon" and "Adrian". Now –
Arthur: Oh! You said you didn't know!
Brian: It really doesn't matter. The point is there were these two servants –
Arthur: He's making it up as he goes along.
Brian: No, I'm not!

Monty Python's *The Life of Brian*

1 Introduction

1.1 Mathematical objects

One of the most central questions in the philosophy of mathematics is whether mathematical objects, e.g., numbers, points, or sets, exist. No matter what answer one gives to this question, it leads into trouble. **Do mathematical objects exist?**

If mathematical objects *exist* then what are they? Presumably they are some kind of abstract, nonphysical objects. This means trouble for at least three reasons. (I) **Yes:** → problems:

(1) How can we refer to such objects? We can't point at them, and it is not at all obvious how we could characterize them in terms of more familiar entities. (1) semantical

(2) How can we obtain knowledge about abstract, nonphysical objects? We can't observe them or have some other kind of causal contact with them, and even if we learn about them by pure reasoning some amount of prior knowledge about them seems to be a prerequisite. (2) epistemological

(3) Why is knowledge about these abstract, nonphysical objects useful with regard (3) practical

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> to concrete, physical objects, and how does its application to them work? We can apply knowledge about one physical object to another in so far as the latter is similar to the former in a certain regard, so that what is true of the one is approximately true of the other; but mathematical objects don't seem to be similar to concrete, physical objects in any regard.

So, if you believe in mathematical objects – in other words, if you adhere to realism in ontology (Shapiro 1997, 3–4, 37) with respect to mathematics – you have three difficult questions to answer.

(II) **No:** → maths is about …

Suppose, on the other hand, that you are an antirealist in ontology, i. e., you *deny* the existence of mathematical objects. Now, mathematical objects are what mathematics appears to be about; so if there are no mathematical objects then either mathematics doesn't have any content, or it has content but we can't take its sentences at face value, as plainly talking about mathematical objects. In the first case, "$2+2=4$" and "$2+2=5$" would both be meaningless, or else both trivially false; in the second case, the one would be true and the other false, but not because they are true or false of objects called "2", "4", and "5". The first, no-content possibility is extremely hard to swallow: the widespread use of mathematics in everyday life and in the sciences strongly indicates that it must contain useful information about something. That leaves the second possibility: mathematics is indeed about something but not about what it seems to pertain to *prima facie*. Adherents of this possibility – realists in truth-value (ibid., pp. 4, 37) who are antirealists in ontology – have to produce plausible alternative candidates for the actual subject-matter of mathematics which are not mathematical objects. Furthermore they should give an explanation for why mathematicians didn't realize themselves what they are in fact talking about, or else if they did, why they don't refer to it outright. And finally, if the subject-matter they propose is abstract and nonphysical too, these antirealists in ontology still have to answer the three questions realists are faced with, only now with regard to their own proposed mathematical subject-matter: how can we talk about it, how can we know about it, and how can this knowledge be usefully applied? Thus, antirealists in ontology who don't cleave to the forlorn position that mathematics is all empty formulas are confronted with tasks similar to those of the realist.

… nothing???

… something else

about what, then?

mathematicians stupid?

inherit the realists' problems

1.2 Structuralism

Structuralism

One comparatively recent approach to these problems is structuralism, which can be characterized by the motto, *Mathematics is the science of structure*. Structuralism is based on the insight that what matters in a mathematical theory is not the intrinsic nature of the objects it appears to deal with but rather their interrelations. As an example, let's look at the natural numbers (0, 1, 2, …). We don't seem to know exactly what they are – if they exist at all – but whatever they are, we know quite well how they are structured: how they are interrelated by relations like successor and less-than, and by functions like addition and multiplication. (E. g., the number 3 is the successor of 2, and the sum of both is greater than 4.) According to structuralism, this way of being structured – this structure – is all that matters in number theory, it is what number theory is about.

structure is what matters

e. g., the numbers

good: answers the practical question

This conception of structures[1] enables the structuralist to give a very convincing answer to the question about the applicability of mathematics: If a mathematical

[1] Michael Resnik (1997) prefers the term "pattern", but I will mostly stick to the terminology Stewart Shapiro (1997) uses.

theory deals with a certain structure $\mathfrak{S}$, and S is a system of interrelated objects (concrete and physical or otherwise) which has that structure, then the theory can simply by that token be applied to S. The theory describes what is entailed by having the structure $\mathfrak{S}$, and thus it describes structural features of, among others, the system S.

systems have structures

I use the term "system" here in the same way as Shapiro does: as a word for a collection of objects combined (usually) with certain relations and/or functions on that collection, where possibly certain objects from the collection are 'designated'. A system is more or less what in model theory is called a structure, except that collections, relations and functions are not supposed to be sets but rather to be understood in some yet-to-be-precisified intuitive sense.

systems

Number theory, to use this example again, deals with the natural-number structure, which can be described by the Dedekind–Peano axioms:

e. g., $\mathbb{N}$-structure via Peano axioms

- every number has a unique successor;
- no two numbers have the same successor;
- every number except one (which we call "zero") is the successor of a number;
- and if zero has some property Φ, and for every number that has Φ, its successor also has Φ, then every number has Φ.

(We could, but needn't, go on to give axioms for addition, multiplication, less-than.) These axioms describe the natural-number structure, not by referring to *it* and detailing what properties it has, but instead by referring to *numbers* and specifying how these are interrelated by the successor relation. Since they talk as if there are numbers, the axioms seem to presuppose the existence of the natural numbers. But the axioms succeed anyway in describing a structure systems can have, independently of whether the numbers exist or not: a system has the natural-number structure (let's call it "the $\mathbb{N}$-structure" for short) if it has a collection of objects and a two-place relation on that collection such that if one interprets the term "number" as referring to the objects, and the expression "is a successor of" as expressing the relation, the Dedekind–Peano axioms become true.

as-if talk

how axioms describe structure: interpretability

Armed with the understanding of the $\mathbb{N}$-structure conferred on her by these axioms, the mathematician can then go on and elucidate further what is involved in having the $\mathbb{N}$-structure. She does this by producing proofs in the as-if idiom of the axioms, talking as if numbers actually exist:

obtaining knowledge through proofs

> Zero has a unique successor; let's call this number "1". Now if 1 were identical with zero then, since 1 is a successor, zero would be a successor, which the axioms say it isn't; so 1 is distinct from zero. This means that there are at least two different numbers.

Talking like that still doesn't compel her to accept that there actually are numbers. She might hold that she is just using a convenient shorthand for talking about whatever systems there may be that have the $\mathbb{N}$-structure (let's call these "$\mathbb{N}$-systems"). The proof apparently about 'zero' and '1' can, for any given $\mathbb{N}$-system S, be read as follows:

> The zero-object of S has a unique S-successor; let's call this object "the 1-object (of S)". Now if the 1-object were identical with the zero-object then, since the 1-object is an S-successor, the zero-object would be an S-successor, which the axioms (interpreted in S) say it isn't; so the 1-object

is distinct from the zero-object. This means that there are at least two different objects in S's collection of objects.

No wonder mathematicians prefer the shorthand version.

reference: via axioms

This view of mathematical practice paves the way for answers to the initial questions about reference and knowledge: We can specify, and talk about, a particular structure by using the as-if talk of an axiomatic characterization – assuming that we have indeed succeeded in specifying a way systems might be structured, and haven't inadvertently produced a set of postulates that is inconsistent. The requirements for having the structure are encapsulated in the axiom system, so the axiom system gives us basic knowledge about the structure. This knowledge can then be extended by unfolding the logical consequences of the axioms.

knowledge: from axioms and proofs

→ **realism in truth-value** … … and possibly in ontology, too

So far, this sketch of structuralism amounts to a version of realism in truth-value which is noncommittal as regards the existence of mathematical objects. Structures in the sense given may themselves be mathematical objects of sorts, but they are not the intended ones (numbers etc.). Rather, the objects intended would, if they exist, constitute systems *having* certain structures, and thereby allowing face-value readings of the corresponding axioms.

– but not necessarily

Even applying one mathematical theory T' (say, number theory) to a system constituted by 'objects' from another theory, T, (say, set theory) doesn't commit one to acknowledging the existence of these latter objects. Applying T' to a system of T-objects means translating T'-theorems into T-theorems, i. e., reading sentences apparently about T'-objects as being about certain T-objects, but the resulting T-sentences can in turn themselves be understood as mere as-if talk: they appear to refer to T-objects but they are actually just shorthand for talk about the objects of whichever systems have the structure that T deals with.[2]

1.3 Places in structures

mathematical objects as **places in structures**

Having gotten this far is nice, but one might still wish for an interpretation of mathematics which allows a face-value reading of mathematical sentences by providing for mathematical objects. This is what Michael Resnik (*Mathematics as a Science of Patterns*, 1997) and Stewart Shapiro (*Philosophy of Mathematics*, 1997) attempt to deliver by their respective versions of a structuralism that is realist in ontology. On their accounts, mathematical objects are *places in structures*.[3] This idea is a combination of two ingredients, one derived from the 'applicability' of structures, the other from structuralist thinking about mathematical objects.

(1) places are **occupiable**

The first ingredient derives from the relation between structures and systems. Consider a particular $\mathbb{N}$-system S, say, the finite sequences of strokes, " ", "|", "||", "|||" etc., together with "y contains one more stroke than x does" as its successor relation. The system S contains one object that is not the S-successor of any S-object, namely, the empty sequence " ". What is special about this object (in the context of S *qua* $\mathbb{N}$-system) can be expressed in various ways. One is to repeat the original characterization: that it is not an S-successor,[4] i. e., there is no finite sequence x of strokes such that the empty sequence contains one more stroke than x does. Alternatively we can also characterize the empty sequence by comparison with a number: The

e. g., occupying the zero-place

[2] The most prominent example for a type of structuralism that opts for realism in truth-value while denying the existence of mathematical objects is Geoffrey Hellman's *Mathematics without Numbers* (1989).

[3] Or 'positions in patterns', in Resnik's terminology.

[4] In the context of an $\mathbb{N}$-system this is obviously equivalent to its being the *only* non-successor.

empty sequence 'does' in S what zero does in the system of the natural numbers, i.e., it plays in S the '*role*' which zero plays in the system of the numbers, i.e., it is related by S-successorship to the other finite sequences of strokes as zero is related by number-successorship to the other numbers. Or, in yet other words, the empty sequence is in S where zero is in the system of numbers, i.e., it occupies in S the '*place*' or '*position*' which zero occupies in the number system. In a sense, there are two 'places' here: one in S, the other in the number system. In another sense, there is but one place: the same place is occupied by the empty sequence and the number zero, only in different $\mathbb{N}$-systems. This latter sense is what Resnik and Shapiro mean when they talk about places in the natural-number structure.

Since the natural numbers (whether they exist or not) are, as it were, the prototypical occupants of places in the $\mathbb{N}$-structure, the place of unique non-successor-object in $\mathbb{N}$-systems is accordingly called the zero-place, after its prototypical occupant. And then there is the 1-place, which is in every $\mathbb{N}$-system occupied by the successor of the zero-place's occupant, and then the 2-place, and so on. So, when a system S has the $\mathbb{N}$-structure, the objects of S thereby automatically occupy specific places in S, in virtue of being related by S-successorship in the way they are. This is supposed to generalize to all other structures and the ways of being related that they determine. Speaking loosely: like a structure is something capable of being instantiated by systems, so its places are things waiting to be occupied by those systems' objects.

systems having structures make objects occupy places

However, talking about 'specific places', or about 'the places' in a structure $\mathfrak{S}$, is useful only if all systems having $\mathfrak{S}$ must be isomorphic.[5] One might perhaps say that, e.g., 'the group[6] structure' provides places in the manner in which a generous host provides sufficiently many chairs for potential guests: not caring if some will be left unoccupied. But used in this way the term "place" doesn't do any work, because saying that an object x occupies some 'group place' in a given group G cannot in general convey any additional information over and above x's being an object of G. Except for the neutral element there are many different roles objects can play in some group which in other groups are played by no object at all. Furthermore there isn't even an upper limit for the number of such places, since groups can have arbitrarily large cardinalities.

'noncategorical' structures don't have definite places

By contrast, when all instantiations of $\mathfrak{S}$ are isomorphic then an object x in one $\mathfrak{S}$-system S has exactly one 'isomorphic counterpart' in every other $\mathfrak{S}$-system (with respect to particular isomorphisms). These isomorphic images of x all play the same role within the framework of their respective $\mathfrak{S}$-systems, and thus we can talk informatively about 'the places' in $\mathfrak{S}$: in every $\mathfrak{S}$-system it is the same number of places with the same distribution of roles which are occupied by the system's objects.

– 'categorical' structures do

'Structures' in the wide sense that allows their having nonisomorphic instantiations are anyway irrelevant for us in our search for mathematical objects, because mathematical objects are always the (purported) subject-matter of *categorical* theor-

"structures" (and objects): what categorical theories are about

[5]Two systems S, S′ are isomorphic iff there is an isomorphism between them, i.e., a bijection mapping the objects of S onto the objects of S′ which also respects structure, i.e., which is such that the images of S-objects are related in S′ exactly like the corresponding S-objects themselves are related in S.

[6]Roughly speaking, a group is a set of things which you can 'add' and 'subtract' (or 'multiply' and 'divide', if you prefer; these are anyway just manners of speaking). More precisely, a group is a collection of objects together with a two-place composition function $\circ$ on that collection such that the following are true: (a) $(x \circ y) \circ z = x \circ (y \circ z)$ for all objects x, y, z (associativity); (b) there is an object e ('the neutral element') such that (i) $x \circ e = x$ for all x and (ii) for each x, there is an x' (the 'inverse' of x) such that $x \circ x' = e$.

ies, i.e., theories whose models are unique up to isomorphism. Therefore we will use the term "structure" in a narrow sense: two systems have the same structure if and only if they are isomorphic;[7] in other words: mathematical theories have a single structure as their subject-matter only if they are categorical. So there is one natural-number structure and one real-number structure, but there are lots of group structures. Being a group is a type of structure, not a structure.

the numbers might be places in the ℕ-structure …

Under this conception of places in structures, why should we contemplate identifying mathematical objects with places in structures? Theorems which are apparently about the numbers can be applied to arbitrary ℕ-systems, and thus are about the objects of ℕ-systems, in some sense. They are not about these objects in themselves though, but rather about them in the context of their respective systems, i.e., *qua* occupants of particular places in ℕ-systems. Nor do arithmetical theorems deal with particular ℕ-systems and their particular occupants of number places. These particulars are neglected completely; all that matters for application is that one have *some* ℕ-system – and thus occupants for the number places – at all. So, number theory, considered with regard to applicability, can even be construed as being just about the places in the ℕ-structure, as describing what occupying them entails. But if number theory is about the numbers and at the same time about the places in the ℕ-structure then the proposal that the numbers are identical with the places doesn't seem so far-fetched anymore, especially when no other, equally satisfying explication is in sight.

… because number places are what number theory is about

places occupying themselves?

We said that (we can talk as if) the numbers are the prototypical occupants of the places in the ℕ-structure. Doesn't this jar with the possibility of their being identical with those places? Not necessarily. It may seem puzzling that each number place, considered as a number, occupies itself (in the system of the numbers). But this is at least no more puzzling than the concept *concept*'s being itself a concept, which seems quite acceptable.[8] For example, when the structuralist says,

> "the zero-place occupies the zero-place in the ℕ-system consisting of the collection of all the number places together with the adequate successor relation for number places",

he switches perspectives during the sentence. The first occurrence of "the zero-place" treats that place like an object; it is from the 'places-are-objects' perspective, as Shapiro (1997, 10) calls it. The second occurrence of "the zero-place", or rather the expression "… occupies the zero-place in …", features that place as something like a one-over-many; it is from the 'places-are-offices' perspective (ibid.).

(2) **relational essence**

The second ingredient to the concept of a place in a structure stems from the basic structuralist insight: all that matters in mathematics are the interrelations between objects. This suggests that if there are mathematical objects then whatever nature they have must somehow *consist* in their interrelations. Thus one might extend our earlier remarks about number theory on page 2 above to say: all that matters about the numbers is their structure; if you know how they are structured you know what they *are*.[9] What distinguishes, say, the number zero is not a specific colour or shape, but only that it is no number's successor, that it has 1 as its sole successor, that it is distinct from 1, that every number is either identical with it or else the successor of the successor of … of it, and lots of further things.

… of numbers

… of places

But these relational properties are distinctive of the zero-*place*, too, it seems: stand-■

[7] Possible weakenings of this concept of structure will be considered in Subsection 2.6 below.
[8] Though Frege would demur; cf. his "the concept *horse* is not a concept" (Frege 1892, 197/71).
[9] But see Shapiro 2006a, pp. 114–16, for a critique of slogans like these.

ing in these relationships with the other objects of an $\mathbb{N}$-system is the same thing as occupying the zero-place in that system. In general, occupying a particular place in an $\mathfrak{S}$-system just means being related to the other objects of the system in the way that is distinctive of that place.[10] It is therefore even more natural to suggest that places in structures are the mathematical objects we have been looking for.

two delimiting intuitions

So, the concept of a place in a structure is delimited by two intuitions, (a) the 'vertical' *occupiability intuition* and (b) the 'horizontal' *relational-essence intuition*:

vertical: occupiable

(a) A place in a structure can be occupied by objects of systems having that structure.

horizontal: relational essence

(b) The nature or essence of a place in a structure consists in the way it is interrelated with the other places in that structure.

occupiability: $\rightarrow$ bijection between places and objects

These two intuitions have different implications. The notion of places being occupiable by objects suggests in our context (i) that every object of an $\mathfrak{S}$-system occupies an $\mathfrak{S}$-place, (ii) no object can be in two places at the same time, i. e., no object occupies two different $\mathfrak{S}$-places in the same $\mathfrak{S}$-system, (iii) a place that is already occupied cannot be occupied by a further object, i. e., no two objects occupy the same $\mathfrak{S}$-place in the same $\mathfrak{S}$-system, and (iv) in any $\mathfrak{S}$-system, every $\mathfrak{S}$-place is occupied by an object. In other words, "occupies" (with respect to a structure $\mathfrak{S}$ and an $\mathfrak{S}$-system S) is a bijection between the objects of S and the places in $\mathfrak{S}$.

relational essence: $\rightarrow$ occupants of different places play different roles, i. e., are discernible

The notion that places in structures have 'relational essences' can be expressed more clearly by saying that occupying a specific $\mathfrak{S}$-place in an $\mathfrak{S}$-system S is just a matter of being interrelated with the other S-objects via the relations of S in a specific way. What is characteristic about a certain place in $\mathfrak{S}$ are the S-relational properties an object must have to occupy that place in S. This suggests that different places in $\mathfrak{S}$ have different relational essences, i. e., their occupants must play different roles in the system they belong to. – We will see that these two intuitions about places pull in opposing directions (see Subsect. 3.2).

Progress

By introducing the concept of places in structures and proposing it as an explication of mathematical objects we have made some progress. Although we do not yet have a firm conceptual grip on places and might furthermore question whether such things really exist, nevertheless they give us a better understanding of mathematical objects than we had before. This is due to places' being coupled to structures, as we can see by returning to our initial three questions. (1) Reference to places is less mysterious than reference, say, to numbers was, because we can refer to places in a way similar to that in which we refer to structures: we specify a structure by specifying what are the requirements for a system to have it, and we can specify a place by specifying what it takes for an object to occupy that place in such a system. (2) This gives us basic knowledge about places, and we obtain more knowledge by logically deriving implications. (3) Finally, as we can apply knowledge about a structure to any system that instantiates it, by adequate interpretation of the corresponding theory's sentences, so we can apply knowledge about a place to the object occupying that place in a system, by reinterpreting sentences about the place as being about the object, in the context of that system.

Problems:

(A) cross-structure identity?

This is progress, but there are still problems. One is about identities between places from different structures. For example, the natural-number structure is distinct from, say, the real-number structure, but it seems quite appropriate to consider

[10] This characterization runs into problems, however, for structures with nontrivial automorphisms; see page 8 and Subsection 3.3 below.

the natural numbers (places in the $\mathbb{N}$-structure, on our present account) to be at the same time real numbers (places in the $\mathbb{R}$-structure). After all, we wouldn't want to have two distinct zeros. So, are the places in the $\mathbb{N}$-structure at the same time places in the $\mathbb{R}$-structure, and if so, why?

cf. Cæsar problem

This is reminiscent of Frege's Cæsar problem. Frege abstracted from the equivalence relation "F is equinumerous with G"[11] between concepts to obtain the second-order function "the number of F"; the number of a concept F might be characterized as what all concepts equinumerous with F have in common. But this left open certain identity questions concerning numbers. Frege could decide whether the number of some concept F is the number of some other concept G by testing whether F and G are equinumerous, but he didn't have a criterion that determined whether, say, Julius Cæsar is the number of some concept F. Thus, while Frege's first candidate account of numbers could deal with identity questions *internal* to this account, it lacked grounds for excluding unwanted identities like this one, which *leave* the domain of descriptions used in the account. Similarly, we may have some inkling of identity conditions for places of the same structure, but we are at a loss with respect to cross-structure identity of places.

(B) identity of indiscernibles?
e.g., cardinal 2-structure

Another problem is about indiscernible places. Consider the cardinal 2-structure, which consists in a system's having exactly two objects, without any relations on these. Incidentally, the system of erasers on my desk right now (ignoring any relations they stand in) has this structure, i.e., there are exactly two erasers on my desk right now. Each of my two erasers occupies a different place in the cardinal 2-structure, relative to this system. But nothing seems to distinguish the places they occupy from one another, so, by the identity of indiscernibles, they would have to be one and the same place. How can the cardinal 2-structure have two different places when these places have exactly the same 'nature' or 'essence'?

The cardinal 2-structure and the cardinal structures in general are very boring structures, and so one may be tempted to suggest that structuralism might well do without the concept of places in the context of these structures, or even without the structures themselves. But the same problem arises in the context of structures where we would very much like to retain places as mathematical objects. One example is Euclidean geometry: the points of Euclidean space all play exactly the same role, there is nothing to distinguish them; so they too would be in danger of being identified down to a single point. Another example is the complex-number structure: there is nothing to distinguish the imaginary unit i from its conjugate $-i$, taken simply as complex numbers.[12] In general, the problem of indiscernible places arises for any structure whose instantiations have nontrivial automorphisms,[13] because in that case different objects can play the same role, and thus occupy indiscernible places.[14]

e.g., Euclidean geometry

e.g., complex numbers and their conjugates

→ nontrivial automorphisms!

These problems arise because we do not yet understand places in structures well enough. To shed more light on this concept is the main aim of this paper. Towards that aim we will start by taking a look at the concept of structures, because the concept of places depends on the concept of structures (you can't understand places without first understanding structures) and some of the problems with places are rooted in problems with structures.

[11] I.e., there is a one–one relation between the objects that fall under F and those that fall under G.

[12] Aren't they distinguished by $i = \langle 0, 1\rangle \neq \langle 0, -1\rangle = -i$? Only after identification of complex numbers with ordered pairs of reals. But this identification isn't itself part of the complex-number structure.

[13] I.e., isomorphisms of the system onto itself which are not the identity mapping.

[14] The indiscernibility problem is considered by some authors to be a problem for structures as independent entities. I will show that this is wrong.

2 Structures

2.1 Models, and simples vs. composites

There is one sense of "structure" that is not going to be discussed here, viz., the model-theoretic one. In model theory, a structure is a tuple consisting of a set (the structure's domain of objects), possibly functions and/or relations[15] on this domain, and possibly some designated objects from the domain, which together are suitable as interpretations for the function and relation symbols and the individual constants of a given formal language.[16] For example, the 'structure' of the natural numbers could be taken to be the septuple $\langle \mathbb{N}, \mathrm{Succ}, <, +, \times, 0, 1 \rangle$ for a rather rich language of arithmetic. Where necessary, I will refer to structures in this sense as *models*.[17] In our context models are, or at least represent, systems.

not: model-theoretic structures

As for the concept of structures needed by the structuralist, there are two directions one can take in developing it:

two types of structure concepts:

- one can consider structures as *simples*, e.g., as universals or types,

(a) structures-as-simples

- and one can consider them as *composites*, e.g., as 'systems of places' or classes of isomorphic systems.

(b) structures-as-composites

I tend to conceive of structures as universals or, to be more precise, as properties of systems. A universal (or one-over-many), for me, is not a set or a class, nor in any other way reducible to entities of a different kind.[18] Rather, I see universals, and thus structures, as simples in the sense of not being composed of entities of other kinds.[19] But this conception of universals as simples isn't shared by everyone. At least in the case of structures-as-universals it is possible to see them rather as 'archetypes' of the systems exemplifying them, as something like Platonic Forms which are themselves structured as their instantiations are.[20] When I will speak of universals here, I will not intend this view, but it should be kept in mind that this alternative conception exists. The distinction I am really interested in is the one between structures as universals, on the one hand, and structures as systems of places, on the other, and I only introduce the distinction between structures as simples and structures as composites because some readers might want to conflate structures-as-universals and systems of places, conceiving of these universals as composite entities.

universals: simples

– or not?

Before I describe and defend my own proposal for how we should understand structures, I sketch Resnik's and Shapiro's conceptions of structures, and try to show that structuralists shouldn't be satisfied with these conceptions.

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15 In the set-theoretic sense, i.e., sets of ordered pairs (or tuples) of objects.

16 This applies to first-order languages. For second-order languages things are more complicated, but not different in principle.

17 Other possible designations would be "$\mathcal{L}$-interpretation" or "$\mathcal{L}$-structure" (with "$\mathcal{L}$" being a stand-in for some formal language) and "Bourbaki structure".

18 Thus my account will be hostage to an informative account of universals, i.e., to a solution for the problem of universals. This, of course, cannot be delivered in the space of this essay.

19 Conceivably some universals might be considered as somehow composed of other universals, e.g., in ways analogous to the Boolean operations of conjunction, disjunction, negation. Still, this would be a new, technical use of "composed".

20 Shapiro's (1997) conception of structures is like that; see Subsection 2.3 below.

2.2 Resnik's patterns

In Resnik's (1997) case, I have neither room nor competence for a comprehensive critique of his version of structuralism, because that would require taking into account the whole Quine-inspired framework he uses. This inadequacy may devalidate my proposal from Resnik's point of view, but I still hope to sway readers who, like me, have somewhat more conservative intuitions about reference and truth than Resnik does.[21]

structuralist announcements …

Resnik starts with typical structuralist statements: it is a "fundamental misconception" to "think of [the numbers] as objects that can be discussed and known in isolation from the others" (Resnik 1997, 201; quotation emended). Rather, "mathematics is concerned with structures involving mathematical objects and not with the 'internal' nature of the objects themselves" (ibid.). These statements can be read as suggestive of the intuitive one-over-many conception of structures, and thus as promising to furnish us with the increased understanding of the subject-matter of mathematics associated with it. But as he goes on, Resnik unknowingly, or in any case tacitly, relinquishes this important virtue of the structure concept again. Thus he arrives at a notion of structure that leaves mathematical objects just as mysterious as they were before.

… and 'placist' execution

patterns as systems of positions

Resnik writes: "I take a pattern to consist of one or more objects, which I call positions, that stand in various relationships" (ibid., pp. 202–3; quotation emended).[22] He tries to clarify his idea of patterns and positions by appealing to geometrical intuitions:

geometric analogy

> A position is like a geometrical point. It has no distinguishing features other than those it has in virtue of being the particular position it is in the pattern to which it belongs. Thus relative to the equilateral triangle ABC the three points A, B, C can be differentiated, but when considered in isolation they are indistinguishable from each other and any other points. Indeed, considered as an isolated triangle, ABC cannot be differentiated from any other equilateral triangle. Geometry reflects this by focusing on structural relationships, such as congruence and similarity, and reserving claims about the identity of geometric objects for contexts where they are related to other geometrical objects. I transfer this geometric analogy to the various structures studied by mathematics. Within a structure or pattern, positions may be identified or distinguished, since the structure or pattern containing them provides a context for so doing. However, just as in geometry, the premier relationships among patterns are structural ones, namely, structural similarity (pattern congruence and equivalence) and structural containment (pattern occurrence and sub-pattern). (Resnik 1997, 203)

[21] Resnik relies on language-immanent, disquotational theories of reference and truth; see Resnik 1997, 14–30, 243–54.

[22] Cf. Resnik 1981, 532: "a pattern is a complex entity consisting of one or more objects, which I call *positions*, standing in various relationships …" In 1997, Resnik doesn't want to affirm anymore that structures actually exist ("there is no fact of the matter as to whether patterns are mathematical objects or entities of any kind", p. 248). In his framework this should amount to disallowing both quantification over structures and (non)identity statements involving structures, but nevertheless he does quantify over structures and recognizes their standing in certain relationships. This is due to a kind of double standard: When you investigate a particular pattern, all the entities you have to talk about are the positions in this pattern; you do not acknowledge the pattern itself as an entity. When, on the other hand, you investigate the interrelations between patterns, as in model theory or category theory, you treat the patterns themselves as mere positions in some larger pattern; here, patterns are indeed entities, but they aren't really patterns anymore (Resnik 1997, 209–12, 246–50). This reminds one of Frege again: "the concept *horse* is not a concept".

That is, when several patterns are considered at once, what matters are global features of patterns, not local ones involving particular positions.

interrelations between patterns

We should get further help in understanding Resnik's concept of patterns from what he tells us about their possible interrelations. He starts with the notion of pattern-congruence and then defines instantiation and the other relationships (occurrence, sub-pattern, truncation, equivalence) in terms of it (ibid., pp. 204–9). For our purposes, congruence and instantiation are the most relevant. About these, Resnik writes:

pattern-congruence: ≈ isomorphy

"models"?

> *Pattern-congruence* is an equivalence relation whose field I take to include both abstract mathematical structures and arrangements of more concrete objects. Thinking of patterns as models of formal systems, it is the relationship which holds between isomorphic models of formal systems.[23] Consider, for example, a first-order [formal] system S with a single two-place predicate "R" and axioms stating that R is a total ordering. This system has many models—all the total orderings—but they are not all congruent to each other. Only those whose domains have the same cardinality are.[24] The set of numbers from one to ten taken in their natural order and ten pennies stacked on each other taken in order from top to bottom are isomorphic models of the system S; so I count the abstract numerical structure and the stack of pennies as congruent.
>
> When a pattern and an arrangement of so-called concrete objects such as the pennies are congruent then I say that the arrangement *instantiates* the pattern. Instantiation then is a special case of congruence in which the objects "occupying the positions" of a pattern have identifying features over and above those conferred by the arrangements to which they belong. The pennies thus instantiate the one-to-ten pattern. (Resnik 1997, 204; his italics)

instantiation: special case of congruence

criticism

(1) patterns and positions remain mysterious

Why shouldn't we be content with this account? Doesn't it capture and illuminate our intuitions about structures? No, it doesn't, for three reasons. First, patterns and positions are just as mysterious as mathematical objects are, if not more so. Certainly the term "pattern" suggests something we are quite familiar with: patterns as something that can be instantiated by objects (e. g., the pattern a particular spiderweb or Rorschach blot instantiates[25]) or by systems of objects, mathematical or otherwise; patterns as something different objects or systems can have in common: one-over-many patterns. But this is not what Resnik presents us with. We wouldn't say about a pattern in the intuitive sense that it consists of 'positions' or of other objects. What positions does a certain Rorschach blot pattern contain?

intuitively, no 'positions' in patterns . . .

. . . but in *mathematical* patterns?

One might defend Resnik's view by adducing that in philosophy of mathematics we aren't interested in arbitrary patterns but only in the special type of pattern mathematics deals with. Patterns of this type do indeed involve objects, which are perhaps best seen as positions in those patterns.

how to understand patterns & positions:

good: first patterns, then positions

Now, there is a good and a bad way of trying to understand these mathematical patterns and their positions. The good way is to introduce patterns as ones-over-many which can be instantiated by systems of objects; after having done that, one can introduce positions in patterns as whatever is occupied in the relevant sense by a system's objects when that system instantiates a particular pattern. This way is good because we understand one-over-many patterns reasonably well (even though there

[23]Or in our terms: between isomorphic systems/models. Formal systems aren't really needed here.

[24]And not even those, necessarily, if their domains are infinite.

[25]More precisely, there are many different patterns instantiated in such cases, depending on how much detail is abstracted away, and where.

is no informative general account of universals extant yet) and thus have a chance of understanding positions, too.

bad: patterns and positions simultaneously

The bad way consists in introducing patterns as systems of positions, as Resnik does, i.e., introducing the terms "pattern" and "position" simultaneously, hoping that they will clarify each other. This kind of account is bad because either it relies on the one-over-many notion of patterns without admitting it, or else it makes both patterns and positions at least as mysterious as mathematical objects are, because all we are told about 'patterns' and 'positions' is that the former consist of the latter, and that the latter stand in certain relationships and do not have any other features. In the second case we could even put the cart before the horse and use mathematical objects to elucidate what positions in patterns are, rather than vice versa, and thus we would not have made any progress; in the first, we would have obtained a clearer, more explicit, and more elegant exposition by using patterns-as-properties right from the start. Structuralism needs structures-as-universals to be worth its salt, and so does Resnik, as I will show.

positions first???

Characterizing patterns as systems of positions would be sensible if we had an anterior understanding of positions independently of structures. But we don't. Furthermore, this notion of positions as entities in their own right would conflict with one of the two central ideas of structuralism, that the essence of mathematical objects is their interrelations. Despite denouncing the picture of mathematical objects (and thus, of positions) "as objects that can be discussed and known in isolation from the others" (Resnik 1997, 201), Resnik comes dangerously close to countenancing this 'fundamental misconception' himself at one point: on page 212 he proposes "positing a space of positions from which patterns could be 'constructed' as *sui generis* entities" as one possible way in which one might develop a formal theory of patterns.

So, Resnik's characterization of patterns as consisting of positions which stand in various relationships is so far at least as mysterious as mathematical objects are.

(2) geometric analogy: incoherent

However, that patterns consist of positions is only part of Resnik's characterization. He also gives us the geometric analogy: "relative to the equilateral triangle ABC the three points A, B, C can be differentiated, but when considered in isolation they are indistinguishable from each other and any other points" (ibid., p. 203). My second criticism is that this analogy doesn't help, because on either of the two possible readings it is incoherent.

(a) triangle ABC arbitrary?

Normally, when a mathematician talks about 'the equilateral triangle ABC', she will have introduced it first by saying something like, "let ABC be an arbitrary equilateral triangle".[26] By this she would mean: "Let's suppose we have fixed, or have been given, three points – call them 'A', 'B', 'C' – forming the corners of an equilateral triangle." After that she could continue talking about 'the' triangle ABC as if she really had a particular triangle in mind. If that is what Resnik wants to be doing – talk about an arbitrary triangle and omit the "let ABC"-part as understood – then he cannot coherently assert that "when considered in isolation [A, B, C] are indistinguishable from each other and any other points". Talking about the points A, B, C presupposes the (usually imaginary) situation of having somehow fixed which particular points one is talking about, e.g., by sticking stakes in the ground or pins in a map, or by giving their coordinates relative to some coordinate system. This again presupposes having distinguished them first.

(b) triangle ABC archetypical???

But I am afraid Resnik sees himself as doing something else: as talking about

[26]Or else she would have constructed the points A, B, C starting from some other arbitrary posit. This is clearly not what Resnik intends.

the equilateral triangle, a geometric figure which doesn't presuppose a surrounding space. Resnik concedes that "points in the same space ... have adequate identity conditions—such as being on the same lines" (1997, 210). But this, he goes on,

geometry without space ...

> is a consequence of developing geometry as a theory of space. Had geometry developed instead as a theory or collection of theories of figures or shapes, then points might only play their role of marking locations *within* figures without marking locations in a containing space. ... [O]ur geometry would attribute no being to points independently of the figures containing them. (Resnik 1997, 210; his italics)

So Resnik's theory of patterns is modeled on a conception of geometry as a theory of figures *without* a containing space; otherwise points in figures would by definition mark locations in that space. Resnik doesn't elaborate what the imagined theory of figures might look like, but it couldn't be a geometry even in the weakest sense. To be a geometry, a theory must be the theory of *some* kind of space, even if we have long stopped seeing geometry as necessarily a theory of *physical* space.

... is not geometry anymore:

what would the triangle ABC be??

As an example, let's see what is left of 'the equilateral triangle ABC' if we forget about the containing space. What difference is there, then, between the triangle ABC, containing three points A, B, C, and the set $\{A, B, C\}$? Presumably the triangle has sides connecting the corners. So, is it any different from the graph with nodes A, B, C and edges connecting all three nodes? Well, the sides of the triangle would themselves consist of points. Then couldn't we consider it as a topological space with three distinguished points A, B, C? In that case the triangle would be topologically isomorphic (homeomorphic) with the 1-sphere, a circle (with distinguished points). Resnik might counter that the sides of the triangle are straight and rigid, or something to that effect. Let's suppose that Resnik succeeds in specifying plausible meanings for "straight" and "rigid" which make sense without a containing space. He would then have two options: (a) He could try to show that these meanings exclude all the other example systems I have given. Thereby he would demonstrate that his spaceless triangle is indeed something specifically geometrical. But I very much doubt that Resnik can accomplish this task. (b) Alternatively, he could admit that his explications of "straight" and "rigid" do also fit for some of my example systems, and go on to argue that these systems are indeed (pattern-congruent with) the equilateral triangle ABC. In so doing, however, he would drain the concept of equilateral triangles of all its geometrical meaning.

(3) patterns and instantiation *without* universals?

The third flaw of Resnik's account of patterns is that it pretends, but does not succeed, to get by without relying on something supposedly as ontologically dubious as universals are. Positions are conceived to be objects, particulars. Though they are abstract, they are less akin to universals than to concrete particulars.[27] Regarding patterns it is not even true that they are entities (there is no fact of the matter), so presumably it is also not true that they are universals. And still, Resnik might claim, his account succeeds in capturing the one-over-many aspect of patterns even without countenancing universals: that an arrangement of objects instantiates a certain pattern is explicated as the arrangement's being congruent with the pattern. So – does Resnik really get instantiation without universals? Resnik certainly expounds instantiation without explicitly mentioning universals as such. Nevertheless he uses universals in three important ways.

(a) pattern-congruence is a universal

For one thing, what is pattern-congruence if not a universal? Admittedly, Res-

[27] I believe Resnik downplays the abstract–concrete distinction somewhere in his book, but have forgotten where.

nik wants neither to quantify over inter-pattern relations nor to make (non)identity statements concerning them,[28] so on his account he may hold that the congruence relation is not an entity. But even though congruence doesn't actually exist Resnik makes essential use of it, and that is ground enough by my lights to say that his account relies on at least one universal.

(b) pattern-congruence implicitly contains structures-as-universals

What is more, by using pattern-congruence Resnik's account implicitly contains structures-as-universals, namely, as what arrangements or patterns-as-systems must have in common to deserve being called congruent. Again, Resnik would probably say that what his theory doesn't mention doesn't exist in the theory. And again, this isn't what counts for me. Hiding unwanted universals by using equivalence relations, whose own ontological status is de-emphasized, is a common maneuver in ontology (cf. similarity between tropes). It is analogous to somebody describing an equivalence relation on a set but claiming that there are no corresponding equivalence classes: even though he hasn't mentioned them explicitly, he has specified them implicitly.

(c) patterns-as-systems presuppose structures-as-universals

Even if we discount the inter-pattern relations, Resnik's account relies on structures-as-universals in a further important place: What does it mean for the positions of a certain pattern to 'stand in various relationships'? What else can it mean but that the pattern as a system of positions is structured in a certain way, i.e., has a certain structure-as-universal? Only by understanding which structure-as-universal he intends can we understand how he envisages a particular structure-as-system. To say that some pattern or arrangement S is congruent with a pattern P is just a roundabout way of saying that S has the structure-as-universal one has introduced by as-if talk about 'the positions of P'. So, Resnik's structures-as-systems presuppose structures-as-universals.

Resnik's explanation of what structures (patterns) are has been seen to be unsuccessful and more confusing than informative. We now turn to Stewart Shapiro and look what insights he has to offer.

2.3 Shapiro's structures

structures: **universals** or **systems-of-places**?

It seems to me that Shapiro must conceive of structures as something like Platonic Forms, i.e., universals which exemplify themselves by their very nature as universals. (Shapiro confirmed this in personal communication.) On the – even less charitable – only other possible reading, he would vacillate between thinking of structures as (simple) universals and as (composite) systems of places.

universals …

For example, he introduces the natural-number structure as "the pattern common to any system of objects that has a distinguished initial object and a successor relation that satisfies the induction principle" (Shapiro 1997, 5; cf. pp. 72–73). Furthermore, he says that "a structure is a reified 'one-over-many' of sorts" (ibid., p. 9), and: "Because the same structure can be exemplified by more than one system, a structure is a one-over-many" (ibid., p. 84). Another characterization he gives is the following: "A *structure* is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system" (ibid., p. 74; his italics). These formulations suggest that structures are universals and thus simples in the sense described on page 9 above – at least they suggest this to readers like me, who consider Platonic

[28] He may even get around the obvious fact that, for example, pattern-congruence is distinct from pattern-occurrence, by paraphrasing the corresponding sentence in an appropriate way.

Forms as a very unpromising account of universals.

But then he also writes that "in a sense, each structure exemplifies itself. Its places, construed as objects, exemplify the structure" (ibid., p. 89). "In a sense" and "construed as": this is certainly much weaker than saying that each structure exemplifies itself *by its very nature as a structure*, for example. This turn of phrase still leaves open the possibility that structures are universals-as-simples; it just smells slightly of Platonic Forms. Much more revealing, I think, is the following quotation:

. . . or systems of places?

– vacillation

> The best reading of
>
> > "the natural-number structure itself exemplifies the natural-number structure"
>
> is something like
>
> > "the places of the natural-number structure, considered from the places-are-objects perspective, *can be organized into a system*, and this system exemplifies the natural-number structure (whose places are now viewed from the places-are-offices perspective)."
>
> . . . The natural-number structure, *as a system of places*, exemplifies itself. (Shapiro 1997, 101; my italics and displays)

First, Shapiro says that "the places . . . can be organized into a system, and this system exemplifies the natural-number structure". I agree with this part: on one side we have the system of places, a composite entity (presumably not a universal), which we get by supplementing the collection of places with an adequate successor relation; and on the other side we have the structure, a universal (presumably simple), which is exemplified by the system. But then Shapiro speaks of the natural-number structure "as a system of places", which suggests that, to the contrary, the $\mathbb{N}$-structure *is* a system of places, among other things.

Shapiro (1997, 93–95) also presents an axiomatic theory of structures. In this context he adheres unequivocally to the conception of structures as systems of places. For example, he says that "a *structure* has a collection of *places* and a finite collection of functions and relations on those places" (ibid., p. 93; his italics). And: "we stipulate that two structures are identical if they are isomorphic" (ibid.). Both of these phrases make straightforward sense only if one sees structures as systems of places; and nowhere are these expressions qualified in any way, e. g., by mentioning that structures are here merely *considered* or *treated* as systems of places. Indeed, Shapiro writes: "As characterized here, then, each structure is also a system" (ibid., p. 94).

axiomatization: systems of places!

Some further light is thrown on Shapiro's structure concept if one looks at the way he tries to pinpoint the difference between mathematical and nonmathematical structures.

relations of mathematical structures are 'formal'

> One difference between the types of structures concerns the nature of the *relations* between the officeholders of exemplifying systems. Consider . . . the baseball-defense structure. . . . There is an implicit requirement that the player at first base be within a certain distance of first base, the pitcher, and so forth. If not, then it is no baseball defense. In mathematical structures, on the other hand, the relations are all *formal*, or *structural*.[29] The only requirements on the successor relation, for example, are that it be a one-to-one function, that the item in the zero place not be in its range, and that the induction principle hold. No spatiotemporal, mental,

[29]By the way, here one might ask what, in the example of the baseball-defense structure, are the relations whose *non*formality illustrates that structure's nonmathematicity. Knowing next to nothing about baseball, I can't even guess.

> personal, or spiritual properties of any exemplification of the successor function are relevant to its being the successor function. (Shapiro 1997, 98; his italics)

The second half of this passage is a little confusing. It is not immediately clear what is meant by "the successor relation". Just before talking about 'the successor relation, for example', Shapiro says that "[i]n mathematical structures ... the relations are all *formal*, or *structural*". So one might think that 'the successor relation' just is a relation in a certain mathematical structure (considered as a system of places), viz., the ℕ-structure, or the system of the natural numbers. But then Shapiro says that one of the requirements on 'the successor relation' is

> "that the item in the zero place not be in its range".

If he were talking about *the* successor relation, a relation of *the* number system, there would be no question as to what kind of objects are in the relation's range: only numbers are. Therefore he would have referred, not to 'the item in the zero place', but either to plain zero, the number, or perhaps to 'the zero-place' itself, which is, for Shapiro, the number zero under a different guise. Since he refers to 'the item in the zero place', however, he must be talking about the successor relation of an arbitrary system *having* (or 'aspiring' to have) the ℕ-structure.

Furthermore it follows that an 'exemplification of the successor function' must be a pair of objects from the system under consideration. If, alternatively, one reads this expression as referring to the successor function of the system as an 'exemplification' of the structure's successor function, this would be weird in two regards. Firstly, one would then consider the ℕ-structure's (or the number system's) successor relation as exemplifiable by whole relations. But if this relation exists, then what exemplifies it are number pairs, rather. Secondly, it would be weird to talk about 'spatiotemporal, mental, personal, or spiritual properties' of a system's successor relation. Surely this only makes sense in connection with the corresponding relata.

Hence we can reformulate the latter part of the above quotation as follows:

> The only requirements on, e.g., the successor relation *of a given system* S are that it be a one-to-one function, that the item in the zero place of S not be in its range, and that the induction principle hold with respect to S. No spatiotemporal etc. properties of any pair of S-objects exemplifying that successor function are relevant to its being the successor function of S.

Now, to which relation is the predicate "formal" supposed to apply: to the successor relation of S or to that of the ℕ-structure? At the beginning, Shapiro writes that the issue of formality "concerns the nature of the *relations between the officeholders of exemplifying systems*" (my italics). This might seem to suggest that what is formal are *these* relations, the relations of exemplifying systems like S. Thus every ℕ-system's successor relation would be formal. This, however, can't be what Shapiro wants to say; the successor relations of ordinary ℕ-systems may well be ordinary too, involving 'spatiotemporal, mental, personal, or spiritual properties' of the system's objects. Rather, the point Shapiro wants to make is that *the* successor relation, i.e., the successor relation of the ℕ-structure, is formal, in the sense that the requirements on the successor relations of exemplifying systems S do not involve the 'intrinsic natures' of the S-objects but only the other relations of S (if there are any).

a structure's relations are 'completely definable' via each other

Still, the characterization of formality we have arrived at is not yet quite as perspicuous as we might hope for. Shapiro attempts to make it more precise:

> If each relation of a structure can be *completely defined* using only logical terminology and the other objects and relations of the system, then they are all formal in the requisite sense. A slogan might be that formal languages capture formal relations. (Shapiro 1997, 98; my italics)

I suppose that "system" here means "structure-as-system". Shapiro knows that the successor relation of the number system can't be defined using only logical terminology and reference to numbers; so what does he mean by "completely defined"? Shapiro's intentions become clearer when we read his second, more sophisticated characterization of formality (before he gives it he delineates an attempt by Tarski to define what distinguishes *logical* notions, which however need not concern us here):

> ... a relation is *formal* if it can be completely defined in a higher-order language, using only terminology that denotes Tarski-logical notions and the other objects and relations of the system, with the other objects and relations completely defined at the same time. (Shapiro 1997, 99; his italics)

'implicit definitions' as explicit def's of systems of places

The objects and relations of the given system, Shapiro says, can be 'completely defined' simultaneously. But in *explicit* definitions, only one thing at a time is defined. So he must be thinking of a Hilbertian *implicit* definition. In this sense, e.g., the Dedekind–Peano axioms 'define' the natural numbers together with their successor relation. A definition of this kind, however, doesn't really pin down a particular system. It just expresses a certain way for systems to be structured, in this case the $\mathbb{N}$-structure (as a universal). Shapiro seems to be intending that such an implicit definition should at the same time be considered as an *explicit* definition of a particular system of places. And if this 'definition' contains, besides logical vocabulary, only variables or schematic letters, i.e., uninterpreted symbols, for objects and relations, then the relations thus 'defined' are formal.

Now, for one thing, this promotion of implicit definitions to explicitness status is dubious. It seems to be mere stipulation or wishful thinking to suggest that there is one special system that is already 'completely defined' by a given implicit definition, singled out from among the infinitude of systems satisfying that 'definition'. Certainly the natural numbers, if they exist, are *characterized* by the Dedekind–Peano axioms – but only in the sense in which these axioms characterize any $\mathbb{N}$-system whatsoever. The Dedekind–Peano axioms do single out something: not a system – even if it be a system of places – but rather a structure-as-universal.

Furthermore it is not clear how, e.g., the $\mathbb{N}$-structure's not requiring anything 'substantial' of the successor relations of exemplifying systems constitutes a property of one special successor relation, namely, of the successor relation of that structure (considered as a system). Later (p. 23 below) we shall see that the point Shapiro wants to make by talking about the formality of the relations of structures can be made much more clearly without reference to structures-as-systems. Thus the obscure concept of relation formality is uncalled for.

mathematical structures are 'freestanding'

Besides the formality of their relations Shapiro adduces a further distinctive feature of mathematical structures. To illustrate it, he first looks at an example of a nonmathematical structure: the baseball-defense structure. This structure requires its instantiations to consist of "people prepared to play ball. Piles of rocks, infants, and chalk marks are excluded" (Shapiro 1997, 99); i.e., a set of chalk marks cannot be a baseball defense, no matter how they are related. In a mathematical structure, by contrast,

> [e]very office is characterized completely in terms of how its occupant relates to the occupants of the other offices of the structure, and any object can occupy any

> of its places. . . . There are no requirements on the individual items that occupy the places; the requirements are solely on the relations between the items. (Shapiro 1997, 100)

To express this property, Shapiro (1997, 100) says that mathematical structures are *freestanding*. This is doubtless correct; it is what accounts for the universal applicability of mathematical knowledge. *Anything at all* can occupy the places of, say, the $\mathbb{N}$-structure – provided that the accompanying relations have the right extensions. For instance, there are $\mathbb{N}$-systems which have the following initial segment:

> the color purple, the $\mathbb{N}$-structure, Ludwig Wittgenstein, democracy, the collection of all proper classes, the blue eraser on my desk, the zero-place of the $\mathbb{N}$-structure.[30]

$\to$ merely more mysterious mathematical objects!

Now, what are we to think about Shapiro's structure concept? He starts out all right when he introduces structures as something which different systems can have in common – as ones-over-many or universals, that is. This is an informative characterization and a comprehensible notion of structures. (The same is true for the way he motivates the concept of a place in a structure.) But then he throws it all away again by – if sometimes halfheartedly – identifying structures with the corresponding systems of places. If structures are systems of places then they can be universals only in the sense of Platonic Forms – an unprofitable notion. Thus we are cut off from those concepts of structure and place we have a chance of understanding. We are left with two new kinds of mathematical objects: these are 'structures' and 'places' we don't know anything about except for what the structure-theoretical axioms tell us. Shapiro's axiomatic structure theory may conceivably be an interesting mathematical theory,[31] and just possibly it even correctly describes the way structures-as-universals are structured, but it invites again the same type of questions which structuralism originally set out to answer.[32]

2.4 Structures as systems

why not structures as systems of places?

I repeat again and again that structures in the sense structuralism needs must be universals which aren't composed of other entities, like places. Why? What would be so bad about structures conceived as systems, whether we then call them "universals" or not?

self-exemplifying universals? – in general no problem

It is certainly not that I am opposed to universals exemplifying themselves. The concept *concept* falls under itself, and the property of being a property exemplifies itself. I have no qualms with that.[33]

problematic self-exemplification

But while self-exemplification works well for some universals, it doesn't work at all well for most others. Is redness itself red? This would presuppose that we can see

[30] What is presupposed here, of course, is that the designations used all have definite references, that these references are pairwise distinct, and that there are at least countably many further entities. Many parts of this presupposition may be contentious, but they are not at issue at this point in the argument.

[31] Although it seems to be not much more than set theory in a different guise; cf. Shapiro 1997, 96.

[32] This is similar to the fate of set theory: it started out as a convenient way of talking about arbitrary mathematical systems, then it was axiomatized for clarification purposes, and now it is just one more (type of) mathematical structure among others.

[33] I then have to deny of course that there are simple closure properties for these kinds of entities. For example, there is no property "being a property which doesn't exemplify itself", or at least none with unrestricted applicability, because otherwise we get Russell's paradox for properties. So, the analogue of the set-theoretical axiom of separation (or of subsets) doesn't hold for properties. But I'm not interested in axiomatizing properties anyway.

redness, the property, as opposed to seeing the redness of particular objects, where of course what we see is only a particular red object, and we see *that* it is red. These self-exemplification sentences are either false or make little sense. There are things we can say about universals without fretting; for example, I take it that what I have hitherto said about structures-as-universals has at least been found comprehensible by my readers, even if they don't agree with me. But then there are things which only a tiny minority of philosophically minded people would dare to assert in earnest. That redness is red is one of these things: it is rather unclear what is being said with such a statement; we don't know what to do with it, except book it as a phrase philosopher X is apt to utter.

Now, is self-exemplification possible in the case of structures-as-universals? If it were, sentences of similar awkwardness would have to be true. The problem isn't their exemplifying themselves, but rather that in order to do so they would have to be systems and properties at the same time. This can't be. Suppose the $\mathbb{N}$-structure – a certain structural property or way of being structured – is also the system of the natural-number places, or of the natural numbers or what have you. Then because the (property of having the) $\mathbb{N}$-structure is that system, the property has a collection of objects and a relation on those objects. An awkward proposition. Also, because the system is a structure-as-universal, it can be exemplified by other systems, it is something systems can have in common. What does it mean to exemplify a system or to have it in common? More awkward propositions. While other, more commonplace systems cannot themselves be exemplified, in this case exemplification of the system presumably consists in having that system's structure, i.e., being isomorphic to it. So, the system, we said, is identical with a certain structural property, and this property, we now see, consists in being isomorphic to that very system. To say that a *system* is identical[34] with the property of being *isomorphic* to that system is very awkward indeed.

structures exemplifying themselves: both universals *and* systems???

a system can't be the property of being *isomorphic* to that system

But, one might ask, is this any weirder than identifying a structure with the property of *having* that structure? Yes, it is. In the case of "structure" and "having" we are dealing merely with "to have" as an auxiliary verb necessary for making the step from designating a property (say, by "the $\mathbb{N}$-structure") to expressing that property (to formulate sentences of the form "S has the $\mathbb{N}$-structure"; cf. Kneale and Kneale 1962, Sect. X.1). The term "isomorphic", by contrast, accomplishes much more than this in the step from "the system of number places" to "being isomorphic to the system of number places". To depict it as a mere linguistic tool for creating an expression from the designation is to strip both "isomorphic" and "system" of their ordinary meanings.

– but a structure can be the property of *having* that structure?

yes: only an auxiliary verb

So, by identifying a structure-as-universal with a system S, one abandons the distinction between S itself and isomorphism to S. Taken literally, that is nonsense. I suppose one *can* talk like that, but what this really amounts to is that one misleadingly uses "the system S" to refer to a property in some contexts, and to a more-or-less-well-specified system, in others. Since we, on the other hand, want to keep our terms reasonably clear, we have to say that structures-as-universals *cannot be* systems, not even systems which are somehow prototypical exemplars of those struc-

let's not use "system" ambiguously

[34] Is *having the* $\mathbb{N}$*-structure* really the same property as *being isomorphic to the number system*? Or, with Frege: do the expressions "has the $\mathbb{N}$-structure" and "is isomorphic to the number system" have the same sense? Here the intensional character of properties makes itself felt as a problem with identity conditions. I think there aren't any compelling grounds either for answering *yes* or for answering *no*. What is certain though is that these properties are very much alike.

tures.[35]

don't we visualize structures as systems!?

An objection: Don't mathematicians tend to conceive of structures as systems? Can this identification be so far of the mark, then? Certainly we can imagine, e. g., the ℕ-structure as an endless row of empty pedestals or pigeonholes for objects. But no ℕ-system – apart from this imagined system – actually has or exemplifies these pedestals. All it has are objects and a relation on those objects. In imagining a row of 'places for objects' one is only doing the best one can do in imagination: visualizing a system of nondescript objects *having* the ℕ-structure as a representative *for* the abstract structure. This is no different from visualizing a nondescript chimneysweep as a representative for the profession of chimneysweep: it doesn't entail that the profession itself is a nondescript chimneysweep. Even when we describe a structure-as-universal axiomatically we do so by describing a system of nondescript objects, without that structure therefore being a system whose domain actually contains nondescript objects.

yes: because visualization can do no better

why not keep systems of places and dump structures-as-universals?

Accepting that structures-as-universals and systems are different in kind, it might still seem that we can opt for keeping the systems of places and jettisoning the structural properties. Thus we would get the mathematical objects we have been looking for without being burdened by the ontologically dubious universals. Alas, this train of thought is laden with mistakes.

(a) we would only get *mysterious* mathematical objects

Firstly, all the mathematical objects we get this way are mysterious ones, as I have argued (pp. 12, 18 above). But mysterious objects are what we already have. Our primary goal is to understand mathematics, and an account of mathematical objects which is so threadbare that it doesn't further this goal is of no value at all.

(b) we would still need universals

Secondly, reliance on systems of places alone doesn't free us from universals. To get instantiation of structures from there we still need isomorphism relations, and these must be conceived as universals, not classes. Otherwise we would base our structuralism on a presupposed set theory, which we aren't allowed to do because set theory must itself be elucidated by structuralism. And anyway, even if isomorphism were to be explained as a set or class we would still need a universal for membership (in that class).

(c) structuralism needs exemplifiable structures

Finally, structuralism needs structures-as-universals to obtain answers to the big questions; systems of places won't do. Structures-as-universals are abstract entities, but because they are exemplifiable they pave the way for convincing answers to the three questions we started with: A structure-as-universal can be referred to by pointing out instantiations or by describing what an arbitrary system would have to be like to exemplify it. Knowledge about it can be obtained by deduction from this description. And this knowledge can then be applied to any exemplifying system by adequate interpretation of the describing sentences. Systems of places, on the other hand, because they cannot be exemplified, are of no use as a basis for structuralism, even if they weren't mysterious by themselves.

→ places *of*, not *in* a structure!

Because the structures needed by structuralism cannot be systems of places, talk of 'places *in* a structure', which suggests just that, should for clarity's sake be abandoned. From now on I will rather speak of 'the places *of* a structure'.

2.5 Structures as universals

structures: properties of systems

Structures are ways for systems to be structured, they are certain properties systems

[35]For similar *third man*–like arguments against structures exemplifying themselves, see Hand 1993 and Chihara 2004, 67–70.

can have or have not.

Not any old property a system can have is a structural property. Though *containing an apple among their objects* is a property systems can have, it is not a type of structure. The question of what it is that makes properties structural ones, what distinguishes types of structures from nonstructural properties of systems, is interesting and important, but it will not be pursued here, because it is inessential for understanding what places are. I assume that our intuitive grasp of the difference between structural and nonstructural is sufficient for our purposes here.

difference between structural and nonstructural

… irrelevant

A more relevant way of illuminating the structure concept is to ask what *systems*, the candidates for exemplifying a structure, are. On page 2 above, I introduced the ℕ-structure as the way 'the numbers' are interrelated or structured. So, is the system of the numbers just a collection of objects? In that case, *another* system would be another collection of objects, and this system, this collection, would have the same structure as the first if *its* objects were interrelated in the same way, i. e., by the same relation(s) and function(s): successor, addition, multiplication and less-than. If we took, e. g., the collection of (finite) strings over the alphabet "a", "b", …, "y", "z", every string would have to have a unique 'successor', and so on. But what is a 'successor' of a string? Is "ba"'s successor "bb" or rather "baa"? Is "bb" 'less than' "baa", or is it the other way round? We don't know what "successor" etc. mean in the context of strings of letters. Of course we can easily fix some meanings for the words, but as yet they don't have any. Thus it looks as if we didn't know what it means to be structured as the Dedekind–Peano axioms say, because the axioms presuppose the notions of successorship etc. This would mean that we don't really know the structure they characterize.

systems: collections of objects?

→ a string's 'successor'??

In fact, if we assume that the system of the natural numbers is just a collection of certain objects, we misapprehend what the Dedekind–Peano axioms say, and thus, what the structure characterized is. The axioms do not presuppose 'the' successor relation and characterize a structure by saying how the objects in a system have to be interconnected by that relation. Rather, they presuppose that candidate systems have a two-place relation of their own, and then talk about how a system's objects have to be interconnected by the *system's* relation; to do this, they use the word "successor" as a placeholder for that relation, whatever it may be in the particular case. And specifying a nonempty collection of objects is of course not sufficient to specify such a two-place relation on that collection: even extensionally, there is always a multitude of different relations of any arity.

axioms talk about a system's objects *and* relations!

So, a system that aspires to having the ℕ-structure has to have both a collection of objects (its domain) *and* a two-place relation on those objects (and maybe more relations and functions besides, depending on how richly we describe the structure). In general, *structures* are ways for the objects of a domain and corresponding relations and/or functions on this domain to be second-order-interrelated:[36] the objects must be first-order-interrelated by the relations/functions in a particular way. (This is also what isomorphisms – structure-preserving mappings – preserve: not, say, number successorship itself but rather a relationship that obtains between the numbers on the one hand and number successorship on the other.) Accordingly, *systems* have to consist of a domain of objects *together with* corresponding relations and/or functions to fill whatever relation and function slots a structure has.[37]

structure: 2nd-order relation between objects and 1st-order relations

[36] If we consider structures as relations between first-order relations and objects, it might be more precise to call them *mixed*-order relations. Thus structures would in general be what Frege (1891, 29/37–38) called "ungleichstufig".

[37] Actually, matters are not quite as simple as that. Topology, for example, deals with systems whose

systems without relations

Of course, structures need not have relation or function slots. In the degenerate case a structure is just a cardinality of the domain, e. g., the cardinal 2-structure, and the exemplifying systems really are mere collections of objects, bare of any relations or functions.

to specify systems, one must also specify relations

The point that systems in general must comprise relations on their domains, and $\mathbb{N}$-systems in particular must comprise a two-place relation as their successor relation, is one I would have considered obvious. If it is indeed obvious then Shapiro knows it too. So, seeing that he maintains that the number places constitute an $\mathbb{N}$-system (e. g., Shapiro 1997, 101), why does he nowhere specify the corresponding successor relation? Presumably he considers this to be unnecessary because there can't be any question what this relation is, because there is no choice: Shapiro must think that the successor relation on the places of the $\mathbb{N}$-structure is already fixed by their being those places. But how can he think that, when it is clear[38] that there are many different dyadic relations on this collection of objects, even infinitely many such that the resulting system also has the $\mathbb{N}$-structure?[39]

– but Shapiro doesn't specify a successorship relation for number places!?

→ he relies on the places being the numbers

I suppose the structures-as-systems conception is leading Shapiro astray here, by letting him reason as follows: we understand the $\mathbb{N}$-structure sufficiently well (it is what is characterized via the Dedekind–Peano axioms), and by knowing the $\mathbb{N}$-structure we also know the system of the number places (it *is* that structure), and thus we also know the successor relation on the number places (it is a component of the system). But if Shapiro proceeds like this, he misses his goal of improving our understanding of the number system: to reach this goal, it doesn't suffice for him to specify better-understood entities as plausible candidates for being the numbers (here he succeeds, by giving us the places of a structure-as-universal); it is also necessary for him to specify a better-understood *relation* on those entities such that the system as a whole is a plausible candidate for being the system of the numbers. This he doesn't do. Instead he relies on our prior understanding of the number system to make us understand the successor relation he envisages for his system of places: number places are to be correlated in just the way the corresponding numbers are correlated. Shapiro's explication of the numbers turns tail halfway through: he usefully illuminates numbers via the places of a structure, but then idly hopes that successorship for places be illuminated via successorship for numbers.

structures also have *relation* places

The point that systems usually must comprise relations can be stated in a more interesting fashion by saying that structures (can) have places not only for objects, but also for first-order relations and functions. As an example, consider again the $\mathbb{N}$-system S of the sequences of strokes together with "y contains one more stroke than x does" as its successor relation. This successor relation for stroke sequences 'does' in S what number successorship does in the system of the numbers; so these successor relations play the same (second-order) *role* in different $\mathbb{N}$-systems. And perhaps we can also say that the one relationship 'is in' S 'where' the other is in the system of the numbers; the two relationships occupy the same *place* in different $\mathbb{N}$-systems. This latter way of talking seems a little queer, because "place" carries a connotation of "place among *objects*". But since we have already given up the

relations ("open", "neighbourhood of ...") involve not just points but also *sets* of points. And many algebraic types of structures, e. g., vector spaces, seem to involve *two* domains or systems (a field and a group, in the example). Nevertheless, these complications seem technical rather than philosophical ones and will therefore be ignored.

[38]Or is it? Possibly someone might claim that there really is only one relation on the number places which turns them into an $\mathbb{N}$-system. But this would make the number places into entities even *more* mysterious than the numbers.

[39]For any $n \in \mathbb{N}$, switch the n-place and the zero-place to obtain a new ordering of the number places.

term's connotations of geography and space, we might as well give up a little more. Thereby we spare us the introduction of an artificial new term which would do the same work for relations as the term "place" does for objects. Thus I will henceforth distinguish between the *object places* of a structure and its *relation* or *function places*. The $\mathbb{N}$-structure has a relation place for a system's successor relation; the cardinal 2-structure has exactly two object places and no function or relation places at all.

distinguished-object places

Distinguished objects, e. g., the zero and the 1 in a field, can be modeled as zero-place functions or as monadic relations true of exactly one object; therefore I have made almost no mention of them. But what applies to relations and functions also applies to distinguished objects, *mutatis mutandis*, and so we must say that structures can have *distinguished-object places*.

. . . aren't object places

These mustn't be confused with object places *simpliciter*. Certainly they are not *additional* object places: whatever occupies a distinguished-object place is simultaneously already the occupant of an ordinary object place. Neither are distinguished-object places *identical* with particular object places: every ring-with-unit structure has two distinguished-object places, its zero- and its 1-place; but these can be occupied by the same object (in rings with unit whose domain contains only a single object), which isn't possible for nonidentical object places.[40] Also, object places do not automatically constitute distinguished-object places. The difference is that distinguished objects can be picked out, referred to, by means of the structure that has them, whereas one can't do this with the occupants of object places in general. For example, given some $n > 1$, the cardinal n-structure doesn't allow for any particular object to be picked out, but the cardinal n-structure *with a distinguished object* does.

(relation places as relations on places?)

We hope to be able to use the object places of structures as mathematical objects; so perhaps we can extend this strategy to relations and use relation places as the relations on those objects? More concretely, one might consider using the $\mathbb{N}$-structure's successor-relation place as the successor relation for the collection of number places, thereby obtaining an $\mathbb{N}$-system from the $\mathbb{N}$-structure with minimum effort. Later we shall see that systems of mathematical objects cannot be obtained quite so cheaply.

back to formality of relations

Now that we have introduced the notion of a structure's relation places, we can return to Shapiro's conception of the 'formality' of the relations in mathematical structures (p. 16 above). Whereas the baseball-defense structure (or its relations, whatever they may be) involves, among other things, the natures and spatial distances of the occupants of its places, the $\mathbb{N}$-structure's successor relation is supposed to be formal. That is to say, the respective successor relations of systems *exemplifying* the $\mathbb{N}$-structure do not involve any properties of or relations between the objects of those systems, except the other 'arithmetical' ones. Or, in yet other words: the successor relation (of the $\mathbb{N}$-structure, considered as a system) can be 'completely defined' – simultaneously with the numbers and the other arithmetical relations – in an implicit definition which uses only logical and arithmetical terminology: the Dedekind–Peano axioms. Earlier (p. 17), we wondered what exactly the property of being formal is, and what it is a property *of*.

'formality of a structure's relations' is rather about that structure's relation *places*

The relevant property can be described more transparently as a property of the $\mathbb{N}$-structure's successor-relation place. The only requirements this place puts on its potential occupants (the successor relations of candidate $\mathbb{N}$-systems S) are that they correlate the S-objects in the extensionally right way. More precisely, the second-order relationship the successor relation of S must bear to the objects of S in order

[40]However, one might argue that in the case of the one-element ring-with-unit structure its zero-place and its 1-place are identical, and we have two descriptions only because we think of this structure as a special case of the ring-with-unit type of structure.

for S to instantiate the $\mathbb{N}$-structure can be expressed in purely logical terms. This relationship is of course again the property of systems expressed by the Dedekind–Peano axioms, i.e., the $\mathbb{N}$-structure-as-universal:

$$\begin{aligned} & \forall x\, \exists y\, \forall y' \left[Rxy' \leftrightarrow y' = y \right] \\ \wedge\ & \forall x, x', y \left[Rxy \wedge Rx'y \rightarrow x' = x \right] \\ \wedge\ & \forall y \left[y \neq c \leftrightarrow \exists x\colon Rxy \right] \\ \wedge\ & \forall X^{(1)} \Big[Xc \wedge \forall x, y \big(Xx \wedge Rxy \rightarrow Xy \big) \rightarrow \forall x\colon Xx \Big], \end{aligned}$$

where I have used "R" as a schematic letter for a two-place relation – S-successor – and "c" as a schematic letter for an object – the zero-object of S; the superscript in "$X^{(1)}$" means that "X" is to be a one-place relation variable. In ordinary language, we can obviously talk about this property of systems just as well by conceiving it as a relationship the object c must have to R and the rest of the domain. Thus, the 'formality of the $\mathbb{N}$-structure's successor relation' might with equal right be characterized as a property of the $\mathbb{N}$-structure's zero-place. In the end, Shapiro's notion of relation formality comes down to a property of mathematical structures, considered as universals: the requirements they put on systems are of a purely logical nature. This turn of phrase, however, suggests that structures are 'purely logical properties of systems', which in turn suggests properties and relations like, e.g., consistency and entailment. Since these associations would be misleading, it is better to say that structures-as-universals can be *expressed* using purely logical vocabulary.

It isn't surprising, then, to see that 'formality of a structure's relations' is connected to that structure's being freestanding, as Shapiro (1997, 100) observes. Freestandingness of a structure $\mathfrak{S}$ and 'formality of $\mathfrak{S}$'s relations' are in fact almost the same thing, only with the emphasis placed on relations here and on objects there. The reason they aren't exactly the same is that when $\mathfrak{S}$ doesn't have any relation places, there are no 'relations of $\mathfrak{S}$' to place an emphasis on.

Also, the connection to definability becomes clearer: Mathematical structures – the kind of properties of systems that form the subject matter of mathematics – can be expressed in purely logical terms, including schematic letters for the components (relations, functions, designated objects) of candidate systems. And the resulting axiom systems, styled "implicit definitions" by Hilbert, were misconstrued by Shapiro as explicit definitions of systems of places.

2.6 Identity criteria for structures

identity criteria for structures

restricted …

… and unrestricted

The question of what structures-as-universals are can be approached in a further way, namely, by searching for an identity criterion for structures. There is a restricted and an unrestricted version of this goal: in the restricted version, we look for a criterion which determines for arbitrary structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$, say, the respective structures of given systems S_1 and S_2, whether they are identical; in the unrestricted version, we want a criterion which determines identity for arbitrary structures and arbitrary *entities*, say, Julius Cæsar.

restricted identity criteria: isomorphism?

Unrestricted identity criteria for structures will not be addressed here at all. For restricted identity criteria there are different candidates, the most obvious of which is isomorphism of instantiating systems:

(I_0) S_1 and S_2 have the same structure, i.e., $\mathfrak{S}_1$ is identical with $\mathfrak{S}_2$, iff S_1 and S_2 are isomorphic.

But although isomorphism is certainly a sufficient condition for having the same structure, it is very plausible that it is not a necessary condition. This is easily seen by looking at the example of lattices. This type of structure can be characterized in two different ways.[41] On the order-theoretic characterization, a lattice is a non-empty set A of objects together with a (partial) ordering[42] $\leq$ on A such that for all $a, b \in A$ there are infima[43] as well as suprema[44] of $\{a, b\}$. The other way of characterizing lattices is algebraic: here, a lattice is a non-empty set A together with two binary operations $\cap$ and $\cup$ on A which are associative and commutative[45] and satisfy laws of absorption, i. e., $a \cup (a \cap b) = a$ and $a \cap (a \cup b) = a$, for all a, b.

example: lattices

order-theoretic ...

... and algebraic

Prima facie, order-theoretic lattices and algebraic lattices are two different types of structure, having nothing in common: an order-theoretic lattice can never be isomorphic to an algebraic lattice, because the former involves one dyadic relation whereas the latter involves two binary functions. Nevertheless mathematicians consider the resulting theories as just two different ways of characterizing one and the same type of structure. This is because the two theories are *definitionally equivalent*:[46] Let $\mathrm{Th}_{\leq}$ and $\mathrm{Th}_{\cap\cup}$ be the theories given by the axioms for order-theoretic lattices and for algebraic lattices, respectively; and consider the following definitions:

definitional equivalence of theories

$$i = a \cap b \quad \leftrightarrow \quad i \leq a, b \ \wedge \ \forall j \leq a, b\colon\ j \leq i, \qquad (D_{\cap})$$

$$s = a \cup b \quad \leftrightarrow \quad s \geq a, b \ \wedge \ \forall t \geq a, b\colon\ t \geq s; \qquad (D_{\cup})$$

$$a \leq b \quad \leftrightarrow \quad a \cap b = a.^{47} \qquad (D_{\leq})$$

(Here, $a \cap b$ and $a \cup b$ are defined to be the infimum and the supremum of $\{a, b\}$, respectively, whose uniqueness follows from the axioms. Reading "$\cap$" as "infimum" in this way, a must obviously be $\leq$ b iff it is the infimum of the two.) Then, the order-theoretic axioms together with the definitions $(D_{\cap})$ and $(D_{\cup})$ imply both the algebraic axioms (and thus the whole theory $\mathrm{Th}_{\cap\cup}$) and the definition $(D_{\leq})$, and vice versa:

$$\mathrm{Th}_{\leq} + (D_{\cap}) + (D_{\cup}) \ = \ \mathrm{Th}_{\cap\cup} + (D_{\leq}).$$

In other words, there is a theory which is a definitional extension[48] (via $(D_{\cap})$ and $(D_{\cup})$, and via $(D_{\leq})$) of both $\mathrm{Th}_{\leq}$ and $\mathrm{Th}_{\cap\cup}$.

This implies that under the definitions given, there is for every model S_1 of the one theory a corresponding model S_2 of the other which has the same domain and, in a certain sense, the *same structure* (without being isomorphic to S_1): for every state of affairs in S_1 that can be expressed by a formula φ_1 of the one language, there is a corresponding formula φ_2 of the other language such that φ_1 and φ_2 can be shown to be equivalent on the basis of either set of axioms together with its supplementary definition(s); and φ_1 holds in S_1 (under some variable-assignment h) iff φ_2 holds in S_2 (under h).

nonisomorphic systems having the same structure

The relationship which thus obtains between S_1 and S_2 can also be characterized

... because of a common definitional expansion

[41] See, e. g., Davey and Priestley 1990. Boolean algebras, for example, are a special kind of lattices.

[42] I. e., a two-place relation $\leq$ such that for all a, b, c, the following hold: $a \leq a$ (reflexivity), $a \leq b \leq c \rightarrow a \leq c$ (transitivity), $a \leq b \wedge b \leq a \rightarrow a = b$ (antisymmetry).

[43] I. e., greatest lower bounds: objects $i_{ab} \in A$ such that $i_{ab} \leq a$, b and for all $i \in A$, if $i \leq a$, b then $i \leq i_{ab}$.

[44] I. e., least upper bounds, cf. fn. 43.

[45] I. e., $a \cup (b \cup c) = (a \cup b) \cup c$, $a \cap (b \cap c) = (a \cap b) \cap c$, $a \cup b = b \cup a$, and $a \cap b = b \cap a$, for all a, b, c.

[46] See Corcoran 1980, and Wilson 1981 (esp. p. 411), where the term "interdefinable" is used instead. Cf. also Shapiro 1997, 241.

[47] On the right-hand side, "$a \cup b = b$" would do just as well.

[48] This is what Shoenfield (1967, 60) calls "extension by definitions".

as follows: let $S_1 = \langle A, \leq_A \rangle$ and $S_2 = \langle A, \cap_A, \cup_A \rangle$, then $\bar{S} := \langle A, \leq_A, \cap_A, \cup_A \rangle$ is a definitional expansion[49] (via $(D_\cap)$ and $(D_\cup)$, and via $(D_\leq)$) of both S_1 and S_2:[50]

$$\begin{tikzcd} & \langle A, \leq_A, \cap_A, \cup_A \rangle \arrow[dl, dash, "(D_\cap),(D_\cup)"'] \arrow[dr, dash, "(D_\leq)"] & \\ \langle A, \leq_A \rangle & & \langle A, \cap_A, \cup_A \rangle. \end{tikzcd}$$

… because of isomorphic definitional expansions

What particular domain S_1 has is of course wholly irrelevant for its structure. So, for S_1 to have the same structure as some other system S_2, it should suffice if, instead of having a *common* definitional expansion, the two systems have *isomorphic* definitional expansions. This allows for their having different domains – or the same domain but with its objects permuted. Thus an order-theoretic lattice $\langle A, \leq_A \rangle$ and an algebraic one, $\langle B, \cap_B, \cup_B \rangle$, would have the same structure in this sense if the following situation obtained (with "$\cong$" for isomorphism):

$$\begin{tikzcd} \langle A, \leq_A, \cap_A, \cup_A \rangle \arrow[r, dash, "\cong"] \arrow[d, dash, "(D_\cap),(D_\cup)"'] & \langle B, \leq_B, \cap_B, \cup_B \rangle \arrow[d, dash, "(D_\leq)"] \\ \langle A, \leq_A \rangle & \langle B, \cap_B, \cup_B \rangle. \end{tikzcd}$$

structure equivalence

Generalizing from lattices to arbitrary systems, we get the following candidate for a concept of having the same structure (and thus for an identity criterion for structures):

(I_1') S_1 and S_2 have the same structure iff S_1 and S_2 have definitional expansions $\bar{S}_1$ and $\bar{S}_2$ which are isomorphic.

This can be illustrated with the diagram

$$\begin{tikzcd} \bar{S}_1 \arrow[r, dash, "\cong"] \arrow[d, dash, "D_1"'] & \bar{S}_2 \arrow[d, dash, "D_2"] \\ S_1 & S_2, \end{tikzcd}$$

where D_1 and D_2 stand for sets of explicit definitions. This equivalence relation is called *structure equivalence* by Shapiro (1997, 91) and corresponds to Resnik's *pattern equivalence* (1997, 209; 1981, 536).[51] Structure equivalence is, as it were, isomorphism modulo definability. Structure equivalence is of course implied by isomorphism.

$\langle \mathbb{N}, \mathrm{Succ} \rangle$ and $\langle \mathbb{N}, +, 0, 1 \rangle$ and definability in higher-order logic

Structure equivalence is really not just one equivalence relation but rather a whole

---

[49] Or 'expansion by definitions' (Shoenfield 1967, 134).

[50] That $\bar{S}$ is a definitional expansion of S_i corresponds to S_i's being a 'full subsystem' of $\bar{S}$ (cf. Shapiro 1997, 91), and to S_i's being a 'truncation' of $\bar{S}$ containing $\bar{S}$ as a 'sub-pattern' (cf. Resnik 1981, 536; Resnik 1997, 209).

[51] Shapiro gives a characterization which, though equivalent, amounts to the more complicated diagram

$$\begin{tikzcd} & & \bar{T} \arrow[dl, dash, "D_1"'] \arrow[dr, dash, "D_2"] & & \\ S_1 \arrow[r, dash, "\cong"] & T_1 & & T_2 \arrow[r, dash, "\cong"] & S_2. \end{tikzcd}$$

This must also be what Resnik (1997, 209) intends, although as it stands, his definition of pattern equivalence requires that S_1 and S_2 have the same domain (i.e., the same positions).

family of them. Consider the system of the natural numbers with the successor relation, $\langle \mathbb{N}, \mathrm{Succ}\rangle$, and the system of the natural numbers with addition, zero and 1, $\langle \mathbb{N}, +, 0, 1\rangle$. There is a simple definition of successorship on the basis of addition and 1:[52]

$$n = \mathrm{Succ}\, m \quad \leftrightarrow \quad n = m+1, \tag{D_{Succ}}$$

and, going in the other direction, there are easy definitions of zero and 1 on the basis of successorship:

$$n = 0 \quad \leftrightarrow \quad \forall m\colon\ n \neq \mathrm{Succ}\, m, \tag{D_0}$$
$$n = 1 \quad \leftrightarrow \quad n = \mathrm{Succ}\, 0. \tag{D_1}$$

Addition, however, does not admit – at least not in the language of first-order logic – of an explicit definition based on successorship.[53] If, however, we allow ourselves to use second-order logic, we can give the following definition:[54]

addition isn't 1st-order-definable from "successor"

... but it is 2nd-order-definable

$$n = l + m \quad \leftrightarrow \quad \forall X^{(3)} \Big[\forall i\colon Xi0i \ \wedge\ \forall i, j, k\big(Xijk \to Xi\,\mathrm{Succ}\, j\, \mathrm{Succ}\, k\big) \ \to\ Xlmn\Big].^{55} \tag{D_+}$$

So, whereas $\langle \mathbb{N}, \mathrm{Succ}\rangle$ and $\langle \mathbb{N}, +, 0, 1\rangle$ are not structure-equivalent with respect to first-order logic (unlike the two lattices $\langle A, \leq_A\rangle$ and $\langle A, \cap_A, \cup_A\rangle$), they are so with respect to second-order logic. I define two systems S_1 and S_2 to be *n'th-order structure-equivalent* iff they have definitional expansions $\bar{S}_1$ and $\bar{S}_2$ with respect to n'th-order logic which are isomorphic. As isomorphism entails first-order structure equivalence, so n'th-order structure equivalence entails $(n+1)$'th-order structure equivalence. Thus we get a family of successively coarser-grained equivalence relations for systems and candidate identity criteria for structures ($n > 0$):[56]

n'th-order structure equivalence

(I_n) S_1 and S_2 have the same structure iff S_1 and S_2 are n'th-order structure-equivalent.

By accepting structure equivalence, not isomorphism, as the adequate conception of sameness in structure, my suggestion that structures have relation and function places besides their object places (p. 22 above) seems to be cast into doubt. If the lattices $\langle A, \leq_A\rangle$ and $\langle A, \cap_A, \cup_A\rangle$, being first-order structure-equivalent, have the same structure, $\mathfrak{S}$, then does $\mathfrak{S}$ have one relation place or rather two function places? There doesn't seem to be any definite array of relation (and function) places which belongs to $\mathfrak{S}$ itself; rather, each type of systems exemplifying $\mathfrak{S}$ has its own array of relation places: one for order-theoretic lattices isomorphic to $\langle A, \leq_A\rangle$, another for

what about 'the' relation places of a structure?

[52] In what follows, I use "Succ" as a function symbol for the sake of brevity and readability.

[53] Given the particular system $\langle \mathbb{N}, \mathrm{Succ}\rangle$, we can characterize addition implicitly via the first-order recursion conditions $n+0 = n$ and $n+\mathrm{Succ}\, m = \mathrm{Succ}(n+m)$. These do not constitute an implicit definition in the sense of fixing, together with the theory of $\langle \mathbb{N}, \mathrm{Succ}, 0\rangle$, the extension of "+" in arbitrary models. Thus Beth's Theorem cannot be exploited to infer the existence of an explicit definition.

[54] This biconditional says more or less that n is the sum of l and m iff every two-place function X which respects the recursion conditions for addition (see fn. 53) outputs the value n for the arguments l, m. Only the biconditional says this by talking not about two-place functions but rather about the corresponding three-place relations between arguments and values.

[55] Here, the superscript in "$X^{(3)}$" indicates that "X" is to be a three-place variable. One could also use a two-place variable, since l is the only i we need on the right-hand side: $n = l + m \ \leftrightarrow\ \forall X^{(2)}\big[X0l \ \wedge$ $\forall j, k\big(Xjk \to X\mathrm{Succ}\, j\, \mathrm{Succ}\, k\big) \to Xmn\big]$.

[56] Cf. Resnik 1997, Sect. 12.4.

algebraic lattices isomorphic to $\langle A, \cap_A, \cup_A \rangle$, still another for their definitional expansions like $\langle A, \leq_A, \cap_A, \cup_A \rangle$, etc.

a place for every definable relation

But there is another road open as well. If n'th-order structure equivalence is taken as the criterion for sameness in structure then we could also consider $\mathfrak{S}$ as having a relation place for *every* relation definable in n'th-order logic from whatever has been taken as basic in the characterization given for $\mathfrak{S}$. Thus $\mathfrak{S}$, the structure of $\langle A, \leq_A \rangle$, would, over and above its $\leq$-place, have a $\geq$-place to be occupied by the converse of the $\leq$-place's occupant, $\cap$- and $\cup$-places for the corresponding binary operations, and so on. Even the cardinal structures would each have at least one relation place, viz., the one to be occupied by their respective identity relations. If we restrict ourselves to first-order logic then $\langle \mathbb{N}, \mathrm{Succ} \rangle$ and $\langle \mathbb{N}, +, 0, 1 \rangle$ have different structures, with the relation places of the former system's structure constituting, or at least corresponding to, a proper subset of those of the latter system's structure; if, however, we adopt higher-order logic, these differences are nullified.

So, the concepts of relation, function, and distinguished-object places of structures shouldn't be abandoned. One must merely keep in mind that what relation (etc.) places a structure has depends on which notion of structure one employs, i.e., which logic one has chosen; and that there may be more relation places than meet the eye.

which identity criterion to choose?

Which identity criterion for structures is the *right* one? Mathematical practice strongly suggests that criterion (I_0), based on isomorphism, is not it. Also, criterion (I_1), based on first-order structure equivalence, seems still too restrictive. In any case, I see as yet no reason for us to choose. What is important, rather, is that we be clear that these different notions of structure exist, and that, if necessary, we specify which of them we use at any one moment.

additional definable *objects*?

Now, a notion of structure which permits us to add or delete definable relations in a system without thereby changing its structure may seem very liberal. What more could one desire? I believe the structure concept could have still more latitude, namely, it could allow for extensions and truncations of the domain by 'definable' objects.[57] Thus one might have sameness of structure even in the face of fluctuations or wholesale replacements of a system's domain. My motivation for this suggestion stems from ontology. Many superficially conflicting ontological theories (say, realism about universals and trope-theoretical nominalism) seem to me nevertheless to describe the same structure, in some sense. They use different 'domains' of entities (e.g., ordinary particulars, plus either universals or tropes) and different fundamental concepts ('relations'; e.g., exemplification of universals vs. having and similarity of tropes), but in the end the same facts, or the same sentences of ordinary language, must result.[58] My attempts to make these intuitions more precise would however go beyond the scope of the present paper. I merely note that if we should accept such a very broad notion of sameness in structure then the concept of the object places of a structure would have to be relativized or liberalized accordingly, just like the concept of relation places was, consequent upon the adoption of structure equivalence.

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[57]This notion of object definability is not the ordinary one. In the ordinary sense, an object is definable iff it is definable as a distinguished object, i.e., iff a monadic relation true of only this object can be defined. In this sense, zero and 1 are definable in $\langle \mathbb{N}, \mathrm{Succ} \rangle$ (p. 27 above). This doesn't affect the domain since these objects were in it beforehand.

[58]Shapiro (1997, Ch. 7) gives a detailed argument for a similar point with respect to competing positions in the philosophy of mathematics.

3 The places of structures

3.1 Resnik's positions

Resnik thinks that the couple of terms he introduces, "pattern" and "position", is useful in illuminating the nature of mathematical objects, so "position" should be more than just a generic term for mathematical objects. Rather, *position* should be a concept which explicates the structuralist's intuitive conception of mathematical object. So, what is Resnik's concept of positions in patterns?

"position": new concept or mere generic term for math'al objects?

Most of what Resnik (1997) says about positions or mathematical objects falls into one of two categories. The first comprises what a platonist, i.e., realist-in-ontology, structuralist would want to say about mathematical objects anyway: they are "featureless, abstract" (p. 4), "causally inert entities existing outside space and time" (p. 82), they are "atoms, structureless points" (p. 201), "whose identities are fixed only through their relationships to each other" (p. 4). This kind of statements is just a repetition of the basic structuralist picture of mathematical objects and thus does not help to make these ideas any clearer than our ordinary notion of mathematical objects.

(a) positions are abstract etc.

The second category yields a delineation of what mathematics does, and doesn't, say about its objects: "Mathematical objects are incomplete in the sense that we have no answers within or without mathematics to questions of whether the objects one mathematical theory discusses are identical to those another treats; whether, for example, geometrical points are real numbers" (p. 90); "there are … good reasons for not assuming that there are facts of the matter as to whether positions from non-overlapping patterns are identical" (p. 210). So, in the special case where two ('different'?) patterns overlap, i.e., when they are (considered as) nondisjoint subpatterns of one comprehensive pattern, there *are* facts of the matter regarding identities between their respective positions, otherwise not.[59]

(b) positions are 'incomplete objects'

By way of contrast, what Resnik's concept of positions in patterns does not involve is their occupiability. For example, when he says, "Material bodies in various arrangements 'fit' simple patterns, and in so doing they 'fill' the positions of simple mathematical structures" (p. 5), he dares only express it in scare quotes.

scary occupiability

Resnik tries to describe the nature of mathematical objects in a way that doesn't ascribe any unwanted surplus properties to them, i.e., he attributes to them only those properties (including relational properties, especially identities) which belong to them anyway *qua* instances of the respective structure. In this manner he avoids falling prey to Benacerraf's (1965) problem: one cannot level the objection against him that he attributes surplus properties where different surplus properties might be attributed just as well.

no unwanted surplus properties

This is, however, at the same time a further substantial drawback of Resnik's account, besides the unsatisfactoriness of his pattern concept: his concept of a position in a pattern is, in the end, nothing more than the concept of an object in a system, augmented by the instruction that we mustn't ascribe to or deny of such an object anything that isn't part of having the structure which the system is supposed to embody. So, Resnik's 'explanation' of the nature of mathematical objects consists in a decree that beyond what mathematical theories say about their objects there just isn't

incompleteness of objects by decree

[59]Cf. Resnik 1997, Sections 10.3, 10.5. – Resnik's definitions of occurrence (p. 205), sub-pattern (p. 206) and truncation (p. 209) all use identity of positions from 'different' patterns in their definientia. In the light of Resnik's distaste for facts of the matter these definitions would then have to presuppose that the patterns in question overlap in the sense just mentioned.

anything more to be said.

should we believe in **mere posits?**

Why should we accept this account of mathematical objects? Why should we believe in the existence of these 'incomplete' objects? According to Resnik (1997, 5), mathematical objects owe their existence to their having been posited by mathematicians, similarly to physicists' positing certain unobservable objects like the different kinds of elementary particles. Positing by itself, however, doesn't suffice; furthermore the theories which talk about these entities must also be useful and fruitful, if not downright indispensable.

(are numbers like electrons?)

Most of us won't want to deny the existence, e. g., of electrons; so, do we have to accept, by the same token, the existence of incomplete mathematical objects? While the parallel to physics is tempting, it is not convincing. We have good reasons to believe in electrons, because the world behaves like there are electrons: There is stuff that has the effects electrons are stipulated to have, and if this stuff isn't electrons then it is at least very similar to them. So electrons exist, to the best of our knowledge. If, one day, we should obtain evidence that this electrons-like stuff works in ways specifiably different from those of electrons then we will have reason to believe in stuff thus characterized instead of electrons.

(no: for arithmetic, as-if talk suffices)

There is, however, no comparable pressure to believe, e. g., in numbers. Number theory successfully describes the natural-number structure no matter whether there is a distinguished system of objects whose very nature consists in exemplifying it. Benacerraf (1973) argues that it would be convenient if mathematical objects exist, because then we could have a unified account of semantics and truth both for mathematical and nonmathematical discourse, viz., Tarski semantics. Still, that isn't grounds enough to justify brute positing like Resnik's. For example, group theory works fine without positing special objects for every 'categorical' group structure one might wish to investigate. One just says: "Let $\langle G, \circ, e\rangle$ be a group with the following properties: ... ", and then goes on *as if* one had specified particular objects and a particular composition. It is similar for number theory: Even if we don't find entities with the right make-up for being *the* numbers, we can always treat number theory on the model of group theory, placidly talking about 'the numbers' as if we had specific objects in mind. Our results will apply to whatever system has the $\mathbb{N}$-structure, whether there is a system of *the* numbers among these or not. Indeed, the nonexistence of *the* numbers wouldn't make any difference at all for the practice of arithmetic.

(indispensability and reality)

Resnik says that the positing of many kinds of mathematical objects has "proved immensely fruitful for science, technology, and practical life, and doing without them is now (virtually) impossible" (1997, 5). This is true, but by itself this would justify merely the practice of talking *as if* mathematical objects existed, not our taking this talk at face value. If however this kind of talk is so immensely useful it can't be purely for linguistic reasons; it must be because it reflects structural features of the real world (and presumably of all other possible worlds as well). In that case one would have to specify these features, and show that among them are the uniquely convincing candidates for being the numbers. This kind of honest ontological toil is being shirked by Resnik.

is positing just as-if talk?

Now, Resnik might perhaps argue that the positing of particular sorts of objects is really nothing more than the resolute adopting of what I call "as-if talk" about these objects. The answer to this is that as-if talk doesn't have any ontological force, it doesn't make anything exist. So, by generously allowing for as-if talk, I am not implicitly endorsing Resnik's claim that mathematical objects owe their existence to positing.[60]

3.2 Shapiro's places

Shapiro's places: two intuitions

Shapiro's (1997) concept of places in structures has two components, viz., the two intuitions about places I presented in the introduction (on p. 7 above): occupiability and relational essence. He explains occupiability using the familiar example of arithmetic:

occupiability …

> Individual numbers are analogous to particular offices within an organization. We distinguish the office of vice president, for example, from the person who happens to hold that office in a particular year, and we distinguish the white king's bishop from the piece of marble that happens to play that role on a given chess board. In a different game, the very same piece of marble might play another role, such as that of white queen's bishop or, conceivably, black king's rook. Similarly, we can distinguish an object that plays the role of 2 in an exemplification of the natural-number structure from the number itself. The number is the office, the place in the structure. (Shapiro 1997, 77)

And, concerning the relational essence of numbers:

… and relational essence

> The essence of a natural number is its *relations* to other natural numbers. … The essence of 2 is to be the successor of the successor of 0, the predecessor of 3, the first prime, and so on. (Shapiro 1997, 72; his italics)

Shapiro's neglect of occupiability

Shapiro never takes a closer look at what occupiability could mean, or what places *qua* occupiables might be. Neither does the occupiability of places play any role whatsoever in Shapiro's more recent reasonings about places (2006a). While the idea that places are like offices is reiterated there a few times, no use is made of it anywhere to illuminate places, their nature, or their identities.

relational essence: hard to precisify

By contrast, the relational-essence aspect of places is investigated more closely in Shapiro 2006a. He looks at a series of different possible explications:

> Mathematical objects have no non-structural [necessary/mathematical/essential] properties (pp. 114–16),

and, finally,

> A given natural number is uniquely characterized by its relations to other natural numbers, and its other essential properties flow from, or are consequences of, this characterization (p. 116).

He adduces objections against each of these formulations, and in the end admits that the structuralist slogans about the relational essence of mathematical objects just can't be upheld.

what's left of relational essence?

When we give up the slogans, what is left of the relational-essence aspect of places? For one thing, any candidate entities for being the numbers, or even just for being used as numbers, must be accompanied by relations such that the whole forms a system having the $\mathbb{N}$-structure. Without specifying adequate relations, you can't specify (the) numbers. And the relations you specify must of course have the right kind of extension: if your '9' has more than one successor, or none at all, or there is more (or less) than one 'number' which does not have a predecessor, or

[60] See also Collins (1998) about good and bad uses of "positing".

whatever, then the system you have specified is not an ℕ-system, not to mention *the* number system. Furthermore, whatever other properties these entities may have aren't conducive for their being usable as numbers (they might even preclude their being *the* numbers): only the properties and relations which fill the relation places of the ℕ-structure are needed, the rest is negligible or troublesome.

evaluation of Shapiro's place concept

If occupiability and relational essence really fitted together to form one coherent concept, this concept would nicely unite the two central ideas of structuralism: Occupiability of a structure's places is just a manifestation of the structure's one-over-many character. Thus, since structures-as-universals aren't that mysterious, these places are less mysterious than mathematical objects ordinarily are. And since their 'essential' properties are precisely the relevant structural ones, they are at the same time perfect candidates for being a structuralist's mathematical objects.

"office places" and "essence places"

That the two components do fit together, however, must be doubted. To see this, let us isolate them into two separate concepts: On the one hand, we consider *office places*, which are like offices in an organization, or like roles to be played; they are places which are occupiable in some yet-to-be-specified sense. (Thus we might also call them "occupation places".) On the other hand, we consider *relational-essence places* (or "essence places" for short), the 'essential characteristics' of which are their relations to the other essence places of the given structure. For Shapiro, his places in structures are both office and relational-essence places. Occupiability and relational essence are, for him, just different aspects of places, where which of the two is more prominent depends on which orientation you assume: the places-are-offices perspective or the places-are-objects perspective (Shapiro 1997, 82–83).

the essence places *are* the numbers

Actually, the term "relational-essence place" is redundant; we might as well just say "mathematical object" instead. The given characterization of essence places merely encapsulates what we desire mathematical objects to be like, after Benacerraf's "What Numbers Could Not Be" (1965). If the essence places of the ℕ-structure exist, they simply *are* the numbers; and the numbers presumably fit the description of the ℕ-structure's essence places exactly.

'places-are-objects perspective' is actually bipartite:

As long as we aren't convinced that office places and essence places are the same, we should also treat Shapiro's 'places-are-objects perspective' as a pair of two different perspectives rather than a single one. Thus we should discriminate between '*essence*-places-are-objects' and '*office*-places-are-objects'.

office-places-are-objects . . .

We encounter the latter type of perspective when Shapiro writes sentences like in the passage I quoted on page 15 above:

> the places of the natural-number structure, considered from the places-are-objects perspective, can be organized into a system, and this system exemplifies the natural-number structure . . . (Shapiro 1997, 101)

Or, in the same vein:

> Places of structures, considered from the places-are-objects perspective, can occupy places in the same or in different structures. (Ibid., p. 100)

If the places considered as objects here were relational-essence places, these sentences would be trivial: Essence places constitute systems and exemplify structures (or inhabit the corresponding office places) by definition. Nor is it news that essence places (mathematical objects) can occupy places other than their home places, both inside and outside their home structure (like, for instance, the real numbers can be used as a Euclidean line). These sentences were worth being written down because they talk about office-like places – i.e., office places – instead, emphasizing that they can function as office occupants, too. So, "places", here, must be read as "office places"; and the perspective taken here would thus more exactly be called "*office*-places-are-objects".

But there are also examples of a different kind:

. . . vs. '*numbers*-are-objects'

> In the system of the natural numbers, 3 itself plays the 3 role. That is, the number 3, *in the places-are-objects perspective,* occupies the 3 office. (Shapiro 1997, 100; my italics)

> Both the system of finite von Neumann ordinals and the system of Zermelo numerals exemplify the natural-number structure. So do the natural numbers themselves, *qua places-are-objects.* (Ibid., p. 101; my italics)

The 'places' viewed as objects here are numbers and certain sets, i. e., mathematical objects. These are places primarily in the relational-essence, not the occupiability sense. They are essence places of the ℕ-structure and 'the' cumulative-hierarchy structure, respectively. So, the perspective exemplified here would be better named "mathematical-objects-are-objects" or "essence-places-are-objects". I shall however prefer the shorter and more suggestive sobriquet "*numbers-are-objects*".

– two different *subjects* rather than two perspectives

Office-places-are-objects and numbers-are-objects are one and the same perspective only if office places and essence places are the same. The latter thesis, however, must be justified, not presupposed. If it is wrong then we aren't even dealing with two different perspectives on the same family of entities but rather with the consideration of two altogether different families.

office places do *not* have relational essences

One state of affairs is clearly diagnostic of the disputed identity: the office places are the essence places if and only if the office places stand in the mathematical relations which are supposed to correlate essence places. If *the* successor relation is exemplified by anything at all, it is by the numbers, i. e., by the essence places of the ℕ-structure. So, if the *office* places of the ℕ-structure are correlated by the successor relation then they must indeed be identical with its *essence* places.

seems like occupiability *entails* relational essence

Coming to the belief that offices stand in relationships like those their occupants stand in isn't too difficult. On page 7 above I tried to motivate the idea that the places – more precisely, the office places – of structures have relational essences. To that purpose, I wrote about the 'essential properties' of the number zero: "these relational properties are distinctive of the zero-*place*, too." If this is indeed so then relational essence, far from being incompatible with occupiability, rather seems to be entailed by it.[61] *Prima facie*, this sounds plausible: Occupiability surely doesn't mean that a particular office place – e. g., the zero-place – can be occupied by just any object of a given ℕ-system. Quite the contrary; in a way, every office place puts very stringent requirements on its potential occupants. For example, the zero-place's occupant in an ℕ-system S mustn't be an S-successor, and it must be the only S-object with this property. This requirement, furthermore, is a structural property. Doesn't this suggest that office places *have* a relational essence?

Shapiro:
office places *do* stand
in the requisite relationships

That office places at least do actually stand in the requisite structural relationships is indicated by Shapiro's statements as well:

> In [the places-are-objects perspective], we treat positions of a pattern as objects in their own right. When we say that the Speaker presides over the House and that a bishop moves on a diagonal, the terms "Speaker" and "bishop" are singular terms, at least grammatically. Prima facie, they denote the offices themselves,

[61] Of course, there is no conflict between incompatibility and entailment if the concept of office places is inconsistent.

> independent of any objects or people that may occupy the offices. (Shapiro 1997, 10)

Analogously, in pure number talk like "1 is the successor of zero", the terms "1" and "zero" would have to be read as referring to the 1- and zero-'offices', i. e., to the corresponding office places. This would imply that what stands in the successor relation, and, by extension, in the arithmetical relations in general, are indeed the office places of the ℕ-structure.

occupants, not offices, have those relations

This, however, is a misconception. We certainly don't want to say that the *office* of Speaker of the House presides over the House, or that the *role* of bishop moves on a diagonal. Offices do not preside, they are held (by people); and what presides, then, is the person holding the office, not the office itself. Likewise, chess roles do not move, they are played (by material objects); and what moves in a game of chess is the object playing the given role. Similarly, all that is clear about office places in, say, the ℕ-structure is that their *occupants* in a given ℕ-system S must be S-successors of each other (in a certain way), not that the office places themselves have any special successor relation of their own. In general, the requisite structural properties are distinctive of a certain place, not in the sense that this place alone *has* these properties, but that in a given system *this place's occupant* alone has these properties (or, more precisely, stands in the relations occupying the corresponding relation places in that system).

When we say, e. g., "The office of president of the USA has a special seal", we are obviously treating this office as an object in its own right: we look at what this office 'does'; we predicate a certain property of it. This surely is office-places-are-objects. If, analogously, in the sentence "The Speaker presides over the House", the term "the Speaker" denotes the Speaker office (Shapiro 1997, 10; see above), then the sentence "The vice president is president of the Senate" (cf. Shapiro 1997, 83) *identifies* the offices of vice president and of president of the Senate. But these are different offices, which are just occupied by the same person, according to the present US constitution.

So, analysis of nonmathematical discourse doesn't really yield any evidence that office places stand in the relations essence places are said to be correlated by. What is correlated by relations like these are always the offices' occupants. Shapiro comes close to admitting this himself:

> No one would say, for example, that the baseball-defense structure is itself a baseball defense. You cannot play ball with the places of a structure, people are needed. (Shapiro 1997, 101)

But he considers this to apply only to nonmathematical structures, which aren't freestanding (see p. 18 above). Still, ordinarily sentences in the office-places-are-objects perspective do not ascribe the relation terms occurring in them to offices or office places, but rather to arbitrary occupants of those. Just like, in chess, what moves across the board is not the bishop role, so in arithmetic, what is added and multiplied aren't number office places, but the occupants of those places: objects in ℕ-systems, possibly even ℕ–*essence* places.

"successor" meaningless for office places

What would "addition" or "successor" mean for office places, anyway? "Successor" does not make sense for strings of the letters "a", "b", ..., "z" (cf. p. 21 above), and it doesn't make any more sense for the office places of the ℕ-structure, beforehand. You can only credit the opposite if you already presuppose that the ℕ-structure's office places are its essence places. But this would be begging the question, as that identity is to be demonstrated in the first place.

there *are* (lots of) adequate relations on the office places . . .

Certainly, relations on the collection of $\mathbb{N}$–office places do exist, particularly ones which would convert this collection into an $\mathbb{N}$-system. But there are too many of these relations, all equally qualified to be 'the' successor relation on the $\mathbb{N}$-structure's office places.

. . . but none of them is distinguished (ontologically)

Now, maybe you ask yourself: "Isn't one of these relations distinguished? When the n–office place is the number n, isn't it obvious which relation must be the true successor relation?: the zero-place is *no* place's successor, the 1-place is the unique successor of the zero-place, and so on." But this particular relation, whose extension is determined by the stipulation that every $\mathbb{N}$–office place occupy itself, is distinguished only from a pragmatic point of view, not from an ontological one: We want to find a system that's a good candidate for being *the* number system. For this purpose, we want to use the collection of the $\mathbb{N}$-structure's office places. Furthermore we want to supplement this collection with a successor relation such that each number can be identified with the place it occupies in the $\mathbb{N}$-structure. This we can achieve only by letting each number office place occupy itself. Thus we have indeed no choice but to employ the specific relation mentioned. Additionally, this relation is particularly easy to define; it is, as it were, the *canonical* successor relation for the $\mathbb{N}$–office places. This choice, however, is forced merely by our goal of using the number places as numbers, and by economical reasons, not by any metaphysical distinctiveness of the mentioned relation.

Peano axioms indirectly describe a *derived* 'successor relation' on number places

While the Dedekind–Peano axioms do *not* say which number place is a successor of which, nevertheless, interpreting them as predicating certain relations of the number places is possible. Consider the $\mathbb{N}$-structure: The Dedekind–Peano axioms describe it, not by saying: "the $\mathbb{N}$-structure is like this and that", but by ascribing to 'numbers' a particular way of being correlated by 'successor'. This way of being correlated (this second-order relation between objects and relationships) *is* the $\mathbb{N}$-structure. The axioms thus describe it in an indirect manner. Analogously, sentences in the numbers-are-objects perspective, like "1 is the successor of zero", by referring to 'numbers' or to nondescript occupants of places, indirectly characterize the corresponding office places: they say what is involved in (or necessary for) occupying them. In this indirect way, certain properties and relations are ascribed to the $\mathbb{N}$–office places, but 'the successor relation' is not among them.

What there is, is a kind of derivative successor relation which the office places do stand in. We can define, for number places p and q:

q is a *number-office-place successor* of p $\quad :\Leftrightarrow$

for any system S to have the $\mathbb{N}$-structure, the occupant of q in S must be an S-successor of the S-occupant of p.

isn't *this* a distinguished successor relation?

– only pragmatically, not ontologically

At this point, some readers may want to protest again: "Surely you must admit that this is a special, distinguished successor relation on the number office places?" It is special indeed, I would answer, but again only from a pragmatical point of view. Ontologically, any other particular dyadic relation which makes an $\mathbb{N}$-system out of the number places is just as 'special'. For example, we could also define:

q is a *number-office-place successor* of p $\quad :\Leftrightarrow$

for any system S to have the $\mathbb{N}$-structure, the following must hold:

if p's occupant in S is the S-double of some S-object, then q's S-occupant is the S-sum of p's S-occupant and 3_S;

> if p's occupant in S is the S-*successor* of the S-double of some S-object, then q's S-occupant is that S-double itself,[62]

where the 'S-double' of x is of course $x +_S x$ (or $2_S \times_S x$, if you prefer). Pragmatically, this would be objectionable, because the definition is unnecessarily complicated. Ontologically, however, the latter relation is just as good a candidate for being 'the' successor relation on the $\mathbb{N}$–office places as the former is, because the system thus obtained also has the $\mathbb{N}$-structure.

number places do *not* stand in the requisite relationships

So, while there is a relation on the number office places which could comfortably be *used* as a corresponding successor relation, 'the' successor relation on the office places does not exist. *A fortiori*, the office places of the $\mathbb{N}$-structure aren't correlated by 'the' successor relation and thus don't have the structural properties needed for being the $\mathbb{N}$-structure's essence places. Therefore I conclude that the office places of a structure are not its essence places.

essence places are *not* occupiable

Nor is this identification plausible if considered from the other direction, because essence places are not particularly occupiable. Consider the zero–essence place, i. e., the number zero. In what sense can zero be 'occupied' by arbitrary objects? Clearly, if an object a from the domain of system S 'occupies' zero then that is by there being an isomorphism from S onto $\mathbb{N}$ which maps a to zero. 'Occupying' an essence place means being mapped onto it by an isomorphism. But if that is all it takes to be occupiable in the requisite sense – to be an *office* place – then the objects of an arbitrary system S_0 having the right structure are just as well suited to be that structure's office places as are its own essence places: A system S_0 has the structure $\mathfrak{S}$ iff there is an isomorphism from S_0 to the system of $\mathfrak{S}$–essence places. But in that case there is for every $\mathfrak{S}$-system S an isomorphism from S onto S_0, thus making the objects of S 'occupy' (in this watered-down sense) those of S_0. So we might as well consider the objects of S_0 – an arbitrary $\mathfrak{S}$-system – to be the office places of $\mathfrak{S}$, instead of the $\mathfrak{S}$–essence places. If the essence places of $\mathfrak{S}$ and the objects of *any* $\mathfrak{S}$-system whatsoever are equally good candidates for being the $\mathfrak{S}$–office places, then, it seems, the $\mathfrak{S}$–office places are really identical with none of them. (Except, of course, that among those $\mathfrak{S}$-systems are also ones which do indeed have $\mathfrak{S}$–office places as their objects.) Therefore essence places are *not* the corresponding office places.

To summarize: on the one hand, the concept of office places doesn't seem unreasonable so far, and the office places of a structure $\mathfrak{S}$ can be used to obtain a system having that structure; but, on the other hand, they cannot be that structure's relational-essence places, i. e., the $\mathfrak{S}$-objects. Thus we have lost again our most promising candidates for comprehensible mathematical objects.

3.3 Places as roles

A place in a structure might be considered as a '*role*', i. e., as a certain relation between an object and a system having the structure considered. Alas, this conception of places has a fatal problem: there are distinct mathematical objects, e. g., different points in Euclidean space, which nevertheless play identical roles.

[62] What this definition does is that for *even* numbers m, the 'successor' of the m-place is the $(m+3)$-place, while for *odd* numbers m, the m-place's 'successor' is the $(m-1)$-place. This yields the ordering 1-place, 0-place, 3-place, 2-place, 5-place, 4-place, etc.: *pairs* $\langle 2k, 2k+1 \rangle$ keep their position; only their components are switched.

3.4 The argument places of relations

The places of a structure are very similar to the 'places' of ordinary relations.

Being smaller than is a two-place relation. What does this mean? *The Cambridge Dictionary of Philosophy* (Audi 1995, 187) says that, in formal languages, a 'degree' or 'arity' is "a property of predicate and function expressions that determines the number of terms with which the expression is correctly combined to yield a well-formed expression". Being two-place would thus be a *linguistic* property.

Does this mean that with respect to a different (formal) language, *being smaller than* might be a one-place or a seven-place relation? It might be expressed as a one-place relation – if the things it is applied to are ordered pairs. It might also be expressed as a three-place relation, e. g., if two of its arguments are compared as to size, and the third one just doesn't matter. So it might perhaps even be expressed as a seven-place relation, viz., for example, if each of the two objects to be compared is specified via three coordinates, whereby six arguments would be accounted for, and the seventh argument is just 'forgotten' again. But these seem to be just different ways of handling a relation which is most perspicuously conceived – independently of any particular language – as being two-place, because it takes input which, no matter in what way the information is divided up for representation purposes, refers to the sizes of *two* objects.

If the smaller-than relation is best expressed as a two-place relation, whatever the language, then there must be logical or ontological reasons for that rather than linguistic ones. Any agent – including nonlinguistic animals and machines – which can 'apply' the smaller-than relation must be able to accomplish information-processing of a certain type in order to do so. In the case of this relation two entities must be compared, and the result of the comparison must be referred back to these entities, must be usable in dealing with these entities. This is where the number 2 for the arity of "smaller than" comes from, I believe, although I do not yet see clearly how the general idea behind this intuition is to be characterized.

Now, we can conceive of the $\mathbb{N}$-structure as a one-place relation: a property some systems have and others don't. We can also think of it as a two-place relation between domains of objects, on the one hand, and prospective successor relations, on the other. But then we can also see the $\mathbb{N}$-structure as a (second-order) $(\omega+1)$-place relation between countably many objects and one (first-order) two-place relation. In this picture, the object places of the $\mathbb{N}$-structure are the ω-many argument places for objects and the $\mathbb{N}$-structure's relation place is its one argument place for a first-order relation.

If we talk like this, we have to differentiate between occupying a place, as it were, 'successfully', and occupying it *simpliciter*. For example, if we fill two different argument places with the same entity then it doesn't matter what we fill the other argument places with: the $\mathbb{N}$-structure won't be satisfied by this input.

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