Boolean-Valued Sets as Arbitrary Objects

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This article explores the connection between Boolean-valued class models of set theory and the theory of arbitrary objects in roughly Kit Fine's sense of the word. In particular, it explores the hypothesis that the set-theoretic universe as a whole can be seen as an arbitrary entity. According to this view, the set-theoretic universe can be in many different states. These states are structurally like Boolean-valued models, and they contain sets conceived of as variable or arbitrary objects.

> Maybe the following analogy will explain my attitude. We use the standard American ethnic prejudice and status system, as it is generally familiar. So a typical universe of set theory is the parallel of Mr John Smith, the typical American; my typical universe is quite interesting (even pluralistic): it has long intervals where GCH holds, but others in which it is violated badly … This seems no less justifiable than stating that Mr John Smith grew up in upstate New York, got his higher education in California, dropped out from college in his third year, lived in suburbia in the Midwest, is largely of Anglo-Saxon stock with some Irish or Italian grandfather and a shade of hispanic or black blood, with a wife living separately and 2.4 children. 'Come on,' I hear. 'How can you [so] treat … CH? You cannot say somewhere yes, somewhere no!' True, but neither could Mr Smith have 2.4 children, and still the mythical 'normal' American citizen is in a suitable sense a very real one.
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> — Saharon Shelah (2002, p. 5)

1. Introduction

Contemporary philosophy of physics aims to develop metaphysical interpretations of fundamental current physical theories. In philosophy of quantum mechanics, for instance, researchers articulate metaphysical accounts of what the physical world at the micro-level *could be* like given our current quantum-mechanical theories. The aim of this article is to do something similar for set theory. The aim is to articulate a metaphysical view of what the set-theoretic world could be like given our current set-theoretic theories and practices.

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The most important development in set theory since the Second World War is Cohen's discovery of *forcing*, which is an incredibly powerful and flexible technique for producing independence results. There are today two main approaches to forcing. The first approach is called the *forcing poset approach* (see Kunen 1980). The second approach is called the *Boolean-valued approach*. The latter was pioneered by Scott and Solovay (and discovered independently by Vopěnka), and was first described in Scott (1967).[1] The two approaches are for most intents and purposes equivalent.

The Boolean-valued approach, as described in Bell (2005), is centred on the concept of Boolean-valued sets, which are functions into a complete Boolean algebra. Boolean-valued sets have been studied mostly with the aim of proving set-theoretic independence results. Here I want to consider structures of Boolean-valued sets from a metaphysical perspective. I will argue that Boolean-valued sets can be seen as *arbitrary objects* in the sense of Fine (1985) and Horsten (2019). Indeed, Fine himself suggests that arbitrary object theory might be applicable to forcing (Fine 1985, pp. 45–6), although his suggestion has hitherto not been followed up.

I will develop the metaphysical hypothesis that there is a sense in which the set-theoretic universe itself is also an arbitrary entity. On the view that I explore, there is only one mathematical universe. But just as with the elements in it, the set-theoretic universe as a whole is an arbitrary entity. And just as the arbitrary sets in the universe can be in different states, the set-theoretic universe can also be in different states.

In this article, the Boolean-valued approach to forcing is used as a tool to express a metaphysical view. Given the mathematical equivalence of the Boolean algebra approach and the forcing poset approach to set forcing, the metaphysical view that I want to explore can also be expressed using the poset approach; but I will not do so here. For *class forcing*, the two approaches are not mathematically equivalent (see Antos, Friedman and Gitman 2021). In this article, I will mostly ignore class forcing.

The proposal explored in this article is tentative: there are multiple ways in which it can be developed more fully. This is just because at this point I am unsure what the best way is of further fleshing out the proposed view. Also, I will not attempt to argue that it is, all things considered, more plausible than rival proposals; that task is left for a future occasion.

[1] The early history of forcing is described in Moore (1987).

The structure of this article is as follows. First, I review key elements of the theory of arbitrary objects. Then I show how in an obvious way arbitrary objects play a role in the model described in Scott (1967), which is the oldest incarnation of the Boolean-valued models approach to forcing. Then I show how arbitrary object theory can also be used to give a metaphysical interpretation of 'modern' Boolean-valued models.

Some elementary knowledge about set and proper class forcing, and about random variables, is required for reading this article. In what follows, I use the notation used in Bell (2005). In particular, concerning algebraic notions, I denote the join, meet, complementation, and implication operations as $\vee$, $\wedge$, ^{c}, and $\Rightarrow$, respectively, and I denote the top and bottom elements of an algebra as 1 and 0, respectively. Moreover, I use the slightly unusual convention adopted by Bell to refer to Boolean algebras as $B, B', \ldots$

2. Arbitrary objects

An *arbitrary F* is an abstract object that can be in a *state* of being some or other F. We may say that an arbitrary F *coincides* with some F in a state, or takes a certain *value* in some state. So, mathematically, an arbitrary F can be modelled as a function

$$f : \Omega \to F,$$

where Ω is a *state space* and F is a collection of objects.[2] In order to develop a basic feeling for what arbitrary objects are like, we briefly consider a few simple examples.

> *Example* 1. A *fair coin* is an arbitrary object. It is an object that can be in a state where it takes the value heads, and it can be in a state where it takes the value tails.

> *Example* 2. Consider an *arbitrary natural number*. Such an arbitrary object can be in a state where it takes the specific number 3 as value, but it can also be in a state where it takes the specific number 4 as value.

Let us briefly relate this second example back to the modelling proposal above.[3] In Example 2, we can evidently take the value space to be $\mathbb{N}$.

[2] For more information about the conception of arbitrary object that is operative in this article, and the role of the notion of *state* in this conception, see Horsten (2019, ch. 3, esp. §§3.5–3.7).

[3] For more details, see Horsten (2019, §4.1.2).

We want there to be as many states as are needed for there to be fully arbitrary natural numbers, that is, arbitrary natural numbers that can be any specific natural number. Moreover, we do not seem to have reasons to have more states. This means that we can set $|\Omega| = \aleph_0$. Arbitrary natural numbers can then be seen as threads through the matrix $\mathbb{N} \times \mathbb{N}$, that is, functions from $\mathbb{N}$ to $\mathbb{N}$.

Typically, for a collection F, there is more than one arbitrary F.[4] For instance, consider an arbitrary natural number a_1 strictly between 3 and 6. Then there is also another arbitrary natural number strictly between 3 and 6, call it a_2, which in every state differs from a_1. So, for instance, in a state where a_1 takes the value 4, a_2 takes the value 5. This shows that arbitrary Fs can be *correlated* with each other.

It has been argued, by Frege (1979) for instance, that there are no arbitrary objects. This is still the prevailing view. But in the spirit of Fine (1985) and Horsten (2019), I will take arbitrary objects ontologically seriously. The aim of this article is not to argue for this metaphysical stance.

In many cases, the function range of an arbitrary F, when regarded as a function, consists of *specific* objects. For instance, in a state where a_1 coincides with the number 4, it takes a *specific* value. But there are also arbitrary objects that can be in a state of being this or that *arbitrary* object. For instance, an *arbitrary* arbitrary natural number strictly between 3 and 6 can be in a state of being the arbitrary number a_1, but it can also be in a state of being the arbitrary number a_2. Such higher-order arbitrariness will play a role in what follows.

I will also be liberal in not just considering maximally specific state descriptions, also known as (Leibnizian) *possible worlds*. I will also permit as states situations that are less than fully specified: call them *partial states*. We will see later how these partial states can be modelled as *sets* of possible worlds.

3. Forcing and random variables

In early work on Boolean-valued models, *random variables* play an important role. In particular, this is so in the first exposition of the method of Boolean-valued models, Scott's (1967) article, 'A Proof of the Independence of the Continuum Hypothesis'.

[4] At least at one point, Fine held that for every F, there is ultimately no more than one 'independent' arbitrary F (Fine 1983, p. 69). I will not make this assumption here.

Scott starts his construction of Boolean-valued models with a probability triple $\langle \Omega, \mathcal{A}, \mathrm{P} \rangle$, where Ω is a state space, $\mathcal{A}$ is a σ-algebra on Ω, and P is a probability function defined on $\mathcal{A}$. This probability triple is the background to the notion of a *random real* over Ω, where a random real over Ω is a function

$$\xi : \Omega \rightarrow \mathbb{R}$$

that satisfies some measurability constraint.[5] Let $\mathcal{R}$ be the collection of random reals. It is easy to see that $\mathbb{R}$ is canonically embedded in $\mathcal{R}$ (by constant functions).

Scott's aim is, roughly, to construct a Boolean-valued analogue of the classical rank $V_{\omega+2}$, which is the level of the iterative hierarchy where the continuum hypothesis (CH) is decided. In this Boolean-valued model the axioms of set theory, in so far as they describe $V_{\omega+2}$, turn out to be true, whereas CH is false.

The language in which Scott describes the initial transfinite levels of the iterative hierarchy has a type-theoretic flavour.[6] In particular, it contains variables ranging over real numbers, and variables ranging over functions on the reals. The set of natural numbers $\mathbb{N}$ is *defined* in this language as a special collection of reals (Scott 1967, p. 95).

In the resulting model $\mathcal{S}$, the real-number variables range over random reals (as defined above). The function variables range over a set $\mathcal{R}^{\mathcal{R}}$ of functions from $\mathcal{R}$ to $\mathcal{R}$ that meet an extensionality condition (Scott 1967, p. 102).

In the Boolean-valued model $\mathcal{S}$, sentences of the language take values in a complete Boolean algebra B, which is obtained from the Boolean σ-algebra $\mathcal{A}$ by identifying events that differ from each other only by a set of probability 0 (as measured by the probability function P). Moreover, B can be seen to have the countable chain condition, which entails that B is *complete*.

Then Scott chooses Ω in such a way that $\mathcal{S}$ contains *many* random reals that are 'orthogonal' to each other. This ensures that $\mathcal{S} \vDash \neg\mathrm{CH}$, where the consequence relation $\vDash$ is based on the *Boolean-valued* truth relation. In particular, the 'degree' to which two random reals ξ and η coincide according to $\mathcal{S}$ is 'measured' by a Boolean value, that is, an element of B. And such an element of B can *roughly*[7] be taken to be the set of states on which ξ and η coincide.

[5] In particular, it is required that for each $r \in \mathbb{R}$, $\{o \in \Omega : \xi(o) \leq r\}$ is measurable.

[6] But this is not essential for his argument, as Scott himself observes.

[7] That is, up to $P = 0$ difference.

Now $\mathcal{S}$ only verifies the usual set-theoretic axioms as far as $V_{\omega+2}$ goes. But Scott sketches how $\mathcal{S}$ can fairly routinely be extended to a Boolean-valued model of *ZFC* that still makes CH false.

As objects that take values in states, Scott's random reals are *arbitrary objects* in the sense of Fine (1985) and Horsten (2019)[8] (or at least they are modelled in the same way). But the values of *function variables* are *not* natural modellings of arbitrary objects. Going up the hierarchy, functionals, and so on, are also *not* arbitrary objects. This 'non-uniformity' is eliminated in later versions of Boolean-valued model theory, such as Bell (2005), as we will see shortly.

The take-away message is that arbitrary objects have played a role in Boolean-valued models from the start. Random variables in Scott's sense have mostly disappeared from modern treatments of Boolean-valued models,[9] and Scott himself already observed that his method for proving the independence of the continuum hypothesis does not really require them (Scott 1967, p. 110).

4. Boolean-valued sets as arbitrary objects

I will now argue that in more recent versions of Boolean-valued model theory, arbitrary objects play an even more pervasive role, albeit in a somewhat less obvious way. First, Boolean-valued sets and Boolean-valued models are defined. Then we will see how complete Boolean algebras can be seen as algebras of situations. To conclude this section, I will discuss two ways of regarding Boolean-valued sets and Boolean-valued models as arbitrary objects, with the Boolean algebra playing the role of the state space.

4.1. Boolean-valued sets

Let us turn to the contemporary approach to Boolean-valued models, as described in Bell (2005). A Boolean-valued class model $V^{(B)}$ consists of functions

$$u : V^{(B)} \to B,$$

where B is a complete Boolean algebra. $\mathrm{Dom}(u)$ can be seen as the *quasi-elements* of u: the elements of $\mathrm{Dom}(u)$ are elements of u only *to a certain extent*, as measured in the algebra B. Moreover, the elements of $\mathrm{Dom}(u)$ are *themselves* Boolean-valued sets. This is reflected in the

[8] See, in particular, Horsten (2019, ch. 10).

[9] But not entirely: see, for instance, Krajíček (2011).

recursive build-up of the universe $V^{(B)}$ of Boolean-valued sets. Indeed, given a complete Boolean algebra B, the Boolean-valued class model $V^{(B)}$ is formally defined as follows by recursion on α (Bell 2005, p. 21):

Definition 3.

$$V_\alpha^{(B)} = \{x \mid \text{Function}(x) \wedge \text{Ran}(x) \subseteq B \wedge \exists \xi < \alpha : \text{Dom}(x) \subseteq V_\xi^{(B)}\}.$$

This means that, just as with ordinary sets, to each Boolean-valued set $u \in V^{(B)}$, a *rank* is assigned.

A Boolean-valued model $V^{(B)}$ comes with a notion of *truth* in $V^{(B)}$. We start by considering the clauses for identity and elementhood in some more detail, since they form the key components of the truth definition. We define (Bell 2005, p. 23, 1.15):

$$[\![u \in v]\!]^B \equiv \bigvee_{y \in \text{Dom}(v)} (v(y) \wedge [\![y = u]\!]^B). \tag{1}$$

This is what it means for some Boolean-valued set u to be *to some extent* (as measured in B) a member of the Boolean-valued set v. Given extensionality, identity and elementhood are intertwined in set theory: identity constitutively depends on elementhood, and vice versa. This is reflected in the clause for identity of the definition of Boolean-valued truth (Bell 2005, p. 23, 1.16):

$$\begin{aligned}[\![u = v]\!]^B \equiv & \bigwedge_{y \in \text{Dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]^B) \\ & \wedge \bigwedge_{y \in \text{Dom}(u)} (u(y) \Rightarrow [\![y \in v]\!]^B),\end{aligned} \tag{2}$$

where $x \Rightarrow y$ is an abbreviation of $x^c \vee y$ (with ^{c} being the complementation operation of B). These clauses are obtained by taking a standard equivalence and then interpreting it within a Boolean-valued model. Thus for u to belong to v is for something identical to u to belong to v.

The Boolean-valued truth conditions of non-atomic statements are exactly what you would expect (Bell 2005, p. 22), so there is no need to spell them out here.

It is easy to see that all these clauses taken together constitute a proper recursive definition that determines a notion of Boolean-valued truth. From now on we will write $V^{(B)} \vDash \phi$ for $[\![\phi]\!]^B = 1$. The atomic clauses of the truth definition are the non-standard ones. Clause (2) is clearly motivated to make the axiom of extensionality come out true, and one can verify that despite the non-standardness of the atomic clauses of the truth definition, the usual laws of identity come out true.

A little more can be said to motivate the atomic clauses. The clause for the elementhood relation says that the *extent* to which $u \in v$ is obtained by:

1. considering each quasi-element y of v;
2. 'measuring' (in B) the extent to which y coincides with u;
3. 'tempering' this measurement by the extent to which y is a quasi-element of v; and
4. summing the values (in B) thus obtained for each y.

In a similar vein, the clause for the identity relation says that the *extent* to which $u = v$ is obtained by considering the extent to which the quasi-elements y of v are elements of u, and vice versa.

One of the basic facts about Boolean-valued models is that for every complete Boolean algebra B, $V^{(B)} \models ZFC$. As in the case of Scott's models from §3, much of the *mathematical* interest of these models lies in the fact that the Boolean algebra B can be chosen such that $V^{(B)}$ makes statements such as CH false (or true).

4.2. Algebras of situations

I will now argue that Boolean-valued models, as well as all the Boolean-valued sets that they contain, can be viewed as arbitrary objects. We have seen in §2 that arbitrary objects are functions from states to values. Unlike Carnapian state descriptions (Carnap 1956, p. 9), which always give a specification of a *fully determinate* or *Leibnizian* possible world, states typically contain only *partial* information about a possible world. Thus states are *situations* (roughly) in the sense of Barwise and Perry (1983).[10] The fair coin, for example, can be in the state of being heads. This state contains only a partial specification of a possible world: it contains no information, for instance, about the number of planets orbiting the sun. A state can therefore be modelled as a *set* of Leibnizian possible worlds, namely, the set of those Leibnizian possible worlds that are compatible with the information that the state contains.

By the Stone representation theorem, any Boolean algebra B is isomorphic to the algebra of the clopen sets of its associated Stone space $\mathrm{S}(B)$.[11] The elements of B can therefore be seen as sets of points in $\mathrm{S}(B)$, and these points can be taken to be Leibnizian possible worlds. It is

[10] Caveat: I do not adopt the formal machinery of Barwise and Perry's theory of situations here.

[11] The elements of $\mathrm{S}(B)$ are the ultrafilters on B, and the topology on $\mathrm{S}(B)$ is generated by the sets of the form $\{\mathcal{U} \in \mathrm{S}(B) : b \in \mathcal{U}\}$ for $b \in B$. The isomorphism is uniquely determined if B is atomic.

therefore natural to regard the elements of B as *sets* of possible worlds, that is, as states.

A Boolean algebra B also contains information about the *structure* of states. When B is seen as an algebra of states, the join operation expresses union of states ('a or b'); similarly for join and complementation. If in B we have $a < b$, then the state a is a *refinement* or precisification of state b. When we look at a and b as states, that is, as sets (of Leibnizian possible worlds), then this means that for i an isomorphism from B to $\mathrm{S}(B)$, $i(a) \subset i(b)$; that is, situation a excludes more Leibnizian possible worlds than situation b does.

The relation of refinement imposes *natural* structure on the collection of situations. Suppose a is a situation where Mary is drinking coffee, and b is a situation where not only is Mary drinking coffee but John is talking to Mary. Then it is natural to regard b as a situation that 'includes' a but is more detailed than a; situation b *extends* situation a.

The algebra B need not be, and typically is not, *atomic*, where atoms are situations that are (according to the algebra) maximally specific. Indeed, B may contain no Leibnizian possible worlds (maximally determinate states) at all. *Partitions of unity*, that is, maximal anti-chains of B, are then especially significant as collections of mutually exclusive and jointly exhaustive sets of states. In atomless contexts, partitions of unity are the closest counterparts to the set of all Carnapian possible worlds.

Actuality plays no role in the picture. Just as it makes no sense to ask which state the fair coin is *actually* in (heads or tails), there is no state that $V^{(B)}$ is actually in. There are just many states that $V^{(B)}$ *can* be in, and that is all we can say in this case. Certainly the maximally unspecific top element $1 \in B$ should not be seen as the actual world. If B contains no atoms, then there is not even a *candidate* for being the actual (Leibnizian) world.

Since the notion of modality that is at play here does not involve the notion of actuality, it cannot be the familiar notion of metaphysical possibility with which, for example, Kripke (1980) is concerned. Yet in the foundations of mathematics there are independent reasons to believe that a modality without actuality is needed. Set-theoretical potentialists, for instance, claim that there *could have been* more sets. On the face of it, this seems to presuppose there that there is a matter of fact about which sets *actually* exist. But any answer to the question 'Which sets actually exist?' seems to contain a high degree of arbitrariness, and the immediate further question 'Why are there not fewer or more sets than exactly those?' seems very hard to answer. Better to say, then, that for the modality at play, the question of how many sets there *actually* are makes no sense. Of course, more needs to be said about the nature of the modality involved in arbitrary object theory, but I do not pursue this question further here.[12]

The foregoing constitutes the reason why B is a prima facie promising candidate for serving as the state space for the Boolean-valued model $V^{(B)}$, as well as the sets containing it, viewed as an arbitrary objects. Indeed, I will go further, and argue that $V^{(B)}$ and all elements of $V^{(B)}$ can naturally be seen as arbitrary objects. Thus the non-uniformity of Scott's model, where the range of the first-order quantifiers is somehow distinguished, is eliminated.

4.3. *Boolean-valued sets as arbitrary objects*

Given that Boolean-valued sets are functions $u : V^{(B)} \to B$, Boolean-valued sets are arrows that 'point in the wrong direction' to be arbitrary objects. Their *domain*, rather than their range, should be a state space. This problem of the arrows pointing in the wrong direction can be remedied, as we will now see.

We start by defining a function * which takes Boolean-valued sets as arguments and yields arbitrary objects as function values:

Definition 4. For every $u \in V^{(B)}$, $u^*(a) = u_a (\in V^{(B)})$, where u_a is defined as the function with the same domain as u such that $\forall s \in \mathrm{Dom}(u_a), u_a(s) = a \wedge u(s)$.

Since the function $*$ takes *states* (elements of B) as inputs, any u^* is an arbitrary object. Hence the function * transforms the Boolean-valued universe $V^{(B)}$ into a universe $(V^{(B)})^*$ of arbitrary objects.

In order to get a minimal feeling how a function f^* is obtained from a Boolean-valued set f, consider the following simple example.

Example 5. Clearly the function $f = \{\langle \varnothing, 1\rangle\}$ is an element of $V^{(B)}$. Therefore the function

$$f^* = \{\langle b, f_b\rangle \mid b \in B\} = \{\langle b, \langle \varnothing, b \wedge 1\rangle\rangle \mid b \in B\} = \{\langle b, \langle \varnothing, b\rangle\rangle \mid b \in B\}$$

is an element of $(V^{(B)})^*$.

The following simple proposition shows that $V^{(B)}$ and $(V^{(B)})^*$ mutually determine each other.

Proposition 6. For every $u \in V^{(B)}$:

(i) u^* is uniquely determined by u.
(ii) u is uniquely determined by u^*.

[12] More information about the modality at play can be found in Horsten (2019, §3.6).

Proof. Clause (i) follows from Definition 4. Clause (ii) of the proposition follows because $u = u_1$ (where 1 is, as before, the top element of B). □

This means that it makes no difference whether we take $V^{(B)}$ to consist of Boolean-valued sets u or their counterparts u^*. In particular, if we modify the truth definition accordingly (left to the reader), then $(V^{(B)})^*$ assigns to any given formula φ of the language of set theory the same (Boolean-valued) truth-value as $V^{(B)}$ does. So, in particular, if B is a complete Boolean algebra, then we also have that $(V^{(B)})^* \models ZFC$.

Whereas every u^* is an arbitrary object, the elements of the *range* of u^*, that is, the Boolean-valued sets u_a, are generally *not* arbitrary objects, but ordinary elements of $V^{(B)}$. Moreover, it is clear that for each Boolean-valued set $u \in V^{(B)}$, each element of the range of u^* is a Boolean-valued set of the *same rank* as u.

Since for each $u \in V^{(B)}$, the arbitrary object u^* is mathematically equivalent to u, we can also see u as a *second-order* arbitrary object, that is, a function from B to $(V^{(B)})^*$:

Definition 7. For every $u \in V^{(B)}$, $u^\dagger$ is the function from B to $(V^{(B)})^*$ such that for all $a \in B$, $u^\dagger(a) = (u_a)^* (\in (V^{(B)})^*)$.

The operation † then transforms the collection of first-order arbitrary objects $(V^{(B)})^*$ into a collection $(V^{(B)})^\dagger$ of second-order arbitrary objects. We can of course continue in this vein, and arrive at the conclusion that for every n, the Boolean-valued universe $V^{(B)}$ is mathematically equivalent to a collection of n-th order arbitrary objects.

Example 8. In Example 5 we saw that the function $f^* = \{\langle b, \langle \varnothing, b \rangle\rangle \mid b \in B\}$ is an element of $(V^{(B)})^*$. Then according to Definition 7, we have

$$f^\dagger = \{\langle c, (f_c)^* \rangle \mid c \in B\}$$

is an element of $(V^{(B)})^\dagger$, where $(f_c)^* = \{\langle b, (f_c)_b \rangle \mid b \in B\}$.
Here $f_c = \{\langle \varnothing, c \wedge 1 \rangle\} = \{\langle \varnothing, c \rangle\}$, whereby $(f_c)_b = \{\langle \varnothing, b \wedge c \rangle\}$. So

$$f^\dagger = \{\langle c, \{\langle b, \langle \varnothing, b \wedge c \rangle \mid b \in B\}\rangle \mid c \in B\}.$$

For any Boolean algebra B, the *restricted Boolean algebra* B_a consists of all elements of the form $y \wedge a$, with $y \in B$, and which is such that for $x, y \in B_a$, $x \star y$ is the same as $x \star y$ in B for $\star \in \{\wedge, \vee\}$, and x^c in B_a is $x^c \wedge a$ in B (Jech 2006, p. 79). If B has the countable chain condition, then

B_a has it also.[13] If B is a complete Boolean algebra, so is the restricted algebra B_a. Therefore, if B is a complete Boolean algebra, then for any $a \neq 0$, we have $V^{(B_a)} \vDash ZFC$. A structure $V^{(B)}$ might be such that neither CH nor ¬CH is true in it, but that it *could be* in a state where CH is true and it *could be* in a state where CH is false, that is, that B contains states a, b such that $V^{(B_a)} \vDash$ CH and $V^{(B_b)} \vDash \neg$CH. Such a $V^{(B)}$ could function as a toy model of a set-theoretic universe in which neither CH nor ¬CH is true.

Thus both $V^{(B)}$ and Boolean-valued sets u in $V^{(B)}$ are arbitrary objects in the following sense. If the Boolean algebra B is atomless, then as we 'go down' B, the universe $V^{(B)}$ takes a more specific state $V_a^{(B)}$ without ever reaching a maximally specific state. Likewise, as we go down B, a given Boolean-valued set u may take more specific state u_a without this state ever becoming maximally specific.

We have seen that for each $u \in V^{(B)}$, u^* is an arbitrary object, or, to be philosophically more correct, can be modelled as such. Thus we can regard every $u \in V^{(B)}$ as an arbitrary object in the sense of §2. Particularly:

- The u^*s are *total* arbitrary objects (since $\mathrm{Dom}(u^*) = B$).
- The state space B of the u^*s consists mostly of *partial* states (since typically the algebra B will be non-atomic).

The $*$-map can be seen as *factorizing* every Boolean-valued set, in the following sense. For every situation $a \in B$, $u^*(a)$ expresses which Boolean-valued set the set u 'is' in situation a, namely, the Boolean-valued set u_a. The $*$-map can also be seen as factorizing the whole Boolean-valued universe $V^{(B)}$ into states: in a state a, the universe $V^{(B)}$ takes the value $V^{(B_a)}$. It is in this sense, I think, that the following statement of Shelah (as well as the epigraph to this article) can be understood:[14] '"Does CH, i.e., $2^{\aleph_0} = \aleph_1$ hold?" is like "Can a typical American be Catholic?"' Shelah (2003, p. 211).

Given that a state can be regarded as a *situation* (see §4.2), a partition of unity $\{a_i \mid i \in I\}$ in B is a collection of mutually exclusive but jointly exhaustive situations. Now let $\{u_i \mid i \in I\} \subseteq V^{(B)}$ and suppose that $v \in V^{(B)}$ is such that $a_i \leq [\![v = u_i]\!]$ for all $i \in I$. Then $V^{(B)} \vDash v = \Sigma_{i \in I}(a_i \wedge u_i)$ (Bell 2005, p. 34, Problem 1.26(ii)). In our terminology of arbitrary objects, this means that $\{a_i \mid i \in I\}$ forms an

[13] A Boolean algebra satisfies the *countable chain condition* if all its anti-chains are countable.

[14] An anonymous referee has pointed out that this is not the only possible interpretation of these quotes.

exhaustive set of situations such that for each situation a_i, the set v takes the value u_i in that situation, and v itself can be seen as a combination of these situations and values. Every situation a_i can furthermore be seen as a Boolean-valued universe $V^{(B_{a_i})}$. So the $V^{(B_{a_i})}$s form a mutually exclusive but jointly exhaustive set of possible ways the set-theoretic universe can be.

We have seen that Boolean-valued sets can be regarded not just as first-order arbitrary objects but also as higher-order arbitrary objects (Definitions 4 and 7). This suggests that Boolean-valued sets can be *recursively defined*, in the spirit of Definition 3 of the hierarchy of Boolean-valued sets, as follows:

Definition 9. For every $u \in V^{(B)}$, u^* is the function from B to $V^{(B)}$ such that for all $a \in B$, $u^*(a) = (u_a)^*$.

This modification of Definition 4 would be well-formed *if* $(u_a)^*$ is generally of lower rank than u^*. We have seen, however, that this is not the case. In other words, since there occurs no reduction of rank of the non-arbitrary object component of u in this process, the Boolean-valued universe $V^{(B)}$ is not equivalent to a collection that consists of elements that are arbitrary objects 'all the way down';[15] even though Boolean-valued sets can be conceived of as (first-order or higher-order) arbitrary objects, they cannot be conceived of as having *only* further arbitrary objects as components.[16]

There is, however, a way of defining a hierarchy of arbitrary sets recursively from other arbitrary sets, and such that it is equivalent to the hierarchy $V^{(B)}$ of Boolean-valued sets (and therefore also to $(V^{(B)})^*$. This *new* hierarchy, which we call V^B (to be distinguished from $V^{(B)}$!), is defined in stages as follows:

Definition 10.

$$V^B_{\alpha+1} = \{f \mid \text{Function}(f) \wedge \text{Dom}(f) = B \wedge \text{Ran}(f) \subseteq \mathcal{P}(V^B_\alpha) \wedge \forall a, b \in B : a \neq b \Rightarrow f(a) \cap f(b) = \varnothing\};$$

$$V^B_\lambda = \bigcup_{\beta<\lambda} (V^B_\beta) \text{ for } \lambda \text{ a limit ordinal.}$$

[15] Thanks to an anonymous referee for making this point.

[16] In this sense, the elements of $(V^{(B)})^*$ and $(V^{(B)})^\dagger$ differ from the higher-order arbitrary objects described in Horsten (2019, §6.9).

The idea behind this is that at every successor stage, every function from B to sets of arbitrary sets that have already been generated,[17] is generated as a new arbitrary set. This definition clearly allows us to assign an ordinal rank α to each element of V^B.

Now we will define a natural one-to-one mapping from the new hierarchy V^B to the Boolean set hierarchy $V^{(B)}$.

Definition 11. For all $f \in V^B$, and for all $f' \in V^{(B)}$: f' *corresponds to* $f \Leftrightarrow \forall g \in V^B, \forall a \in B$: $g \in f(a) \leftrightarrow f'(g') = a$, with g' corresponding to g.

Observe that this *is* a well-formed recursive definition, since the Rank$(g) <$ Rank(f).

It is then not hard to see that the $'$-mapping defines the desired one-to-one function:

Proposition 12.

(i) For all $f \in V^B$, there is a unique $f' \in V^{(B)}$: f' corresponds to f.
(ii) For all $f' \in V^{(B)}$, there is a unique $f \in V^B$: f' corresponds to f.

Proof. Straightforward simultaneous transfinite induction on ranks. We define f' given f and, conversely, f given f' in a straightforward way using Definition 11, observing that we can define f' from f in this way only because f satisfies the condition $\forall a, b \in B : a \neq b \Rightarrow f(a) \cap f(b) = \varnothing$. □

Example 13. The constant function $c : B \rightarrow V^B$ such that $\forall a \in B$: $c(a) = 0$ is an element of V^B_1. It is easy to see that c' is the empty function, that is, $\varnothing$.

As before, Proposition 12 entails that it makes no mathematical difference whether we work with V^B or with $V^{(B)}$. In particular, if we modify the truth definition accordingly (again left to the reader), then V^B assigns to any given formula φ of the language of set theory the same Boolean-valued truth-value as $V^{(B)}$ does.

5. Kaleidoscopic absolutism

We have seen how Boolean-valued universes, and the sets that they contain, can be seen as arbitrary objects. Now I will argue that the

[17] Except those functions that do not satisfy the requirement $\forall a, b \in B : a \neq b \Rightarrow f(a) \cap f(b) = \varnothing$.

set-theoretic universe as a whole can itself be seen as an arbitrary entity. The slogan is, roughly:

The set-theoretic universe is an *arbitrary* $V^{(B)}$.

Let us designate such an arbitrary $V^{(B)}$ as $\mathcal{V}$.

As in the case of arbitrary natural numbers (§2), $\mathcal{V}$ can be modelled as a function from a state space to a value space. The value range of $\mathcal{V}$ is a collection of $V^{(B)}$s where B is an element of a *collection* $\mathcal{B}$ of complete Boolean algebras. Since (as in Example 2) we do not need more states than there are elements of $\mathcal{B}$, the collection $\mathcal{B}$ can then be taken to be the state space of $\mathcal{V}$. So $\mathcal{V}$ can be modelled as the (possibly proper class-size) function that takes each $B \in \mathcal{B}$ to $V^{(B)}$.

At this point, my account becomes somewhat vague: I am not able to say with much precision what $\mathcal{B}$ and V are like. $\mathcal{B}$ should be a *large* collection of complete Boolean algebras (possibly proper class-sized), so that what set theorists regard as real possibilities are all represented as states that $\mathcal{V}$ can be in. V should be large enough to maximize the interpretative power of set theory (Steel 2014, §5). Beyond this, I see only a few more constraints that $\mathcal{B}$ and V satisfy: see below.

You may ask: can we not 'complete' the state space of $\mathcal{V}$ to a *complete Boolean algebra* $\mathcal{B}^*$ and take the set-theoretic universe to be (structurally like) $V^{(\mathcal{B}^*)}$? But this does not work. As is pointed out, for instance, in Antos, Friedman and Gitman (2021), if $\mathcal{B}$ is indeed a proper class, then $\mathcal{B}^*$ is a hyperclass, and $V^{(\mathcal{B}^*)}$ therefore does not make *ZFC* true.

So on the proposed view, the set-theoretic universe is an arbitrary entity that *can* be structurally like a $V^{(B)}$, and only like some $V^{(B)}$; but this arbitrary entity $\mathcal{V}$ itself *is not* structurally like a $V^{(B)}$. Here the requirement that the set-theoretic universe can *only* be structurally like a $V^{(B)}$ is an expression of the confidence of many set theorists that 'possibilities' only arise in ways described by forcing techniques. Of course, it is not guaranteed that this confidence will turn out to be justified in the long run.

Like all slogans, the motto that the set-theoretic universe is an *arbitrary* $V^{(B)}$ has to be taken with a grain of salt. Because of well-known anti-reductionist arguments (for instance, Benacerraf 1965), the thesis should not be that the set-theoretic universe is an entity that can be in the state of *being* this or that $V^{(B)}$. After all, just as it is unreasonable to hold that the number 19 *is* some pure set or other, so it is unreasonable to maintain that the set-theoretic universe can be in a state of being some $V^{(B)}$. The point is rather that it can be in states that are structurally like, or can be fruitfully *modelled* as, $V^{(B)}$s.

It is sometimes argued that there are different, equally valid concepts of set, and that it is somehow indeterminate which of these notions is described in set theory.[18] The position I am putting forward here is not intended as an articulation of this view. The thought that I am trying to develop is *not* that two states of the set-theoretic universe describe different set *concepts*. Rather, the view is that there is *one* conception of set that the set-theoretic universe answers to: a notion of set as an arbitrary object.

The central component of the proposed view consists of truth definitions for the formulae of the language of set theory ($\mathcal{L}_{ZFC}$). I have sketched the definition of truth in a Boolean-valued structure in §4. But a natural definition of truth in the set-theoretic universe $\mathcal{V}$ can also be given:

$$\mathcal{V} \vDash \varphi \equiv \text{ for all } V^{(B)} \text{ in the value range of } \mathcal{V} : V^{(B)} \vDash \varphi.$$

As adumbrated above, this truth definition is somewhat vague: we do not have a strong grasp of what the range of $\mathcal{V}$ is.

By Tarski's theorem on the undefinability of truth, this definition for truth in $\mathcal{V}$ can only be expressed in an extension $\mathcal{L}^{+}_{ZFC}$ of the language $\mathcal{L}_{ZFC}$. This 'Boolean-valued' truth definition quantifies over Boolean-valued models ($V^{(B)}$s), which consist of Boolean-valued sets that are applied to other Boolean-valued sets, Boolean operations that are applied to values of Boolean-valued sets, and so on. We have seen that every Boolean-valued set can be seen as an *arbitrary object*, and that also the $V^{(B)}$s themselves can in the same sense be seen as arbitrary objects.

The view being proposed is meant to be a *foundational* interpretation. As such, ideally, it stands on its own two legs and is not parasitic on any other foundational interpretations, in particular standard interpretations of set theory exclusively in terms of '$\{0, 1\}$-valued sets'. The proposed view should be *logically and conceptually autonomous* from them, in the sense that it should be possible to *state* it without appealing to notions belonging specifically to such interpretations, and that it should be possible to *understand* it without first understanding the notion of $\{0, 1\}$-valued set (Linnebo and Pettigrew 2011, p. 241). To what extent does the present proposal satisfy this requirement?

We have formulated *two* hierarchies of arbitrary sets that are equivalent to the hierarchy $V^{(B)}$ of Boolean-valued sets: $(V^{(B)})^*$ and V^B. Boolean-valued sets are *components* of the arbitrary sets belonging to $(V^{(B)})^*$. Moreover, the hierarchy $V^{(B)}$ of Boolean-valued sets is defined

[18] See, for instance, Hamkins (2012, p. 416), but also Nodelman and Zalta (2014).

in ordinary set theory. So, arguably, universes of the form $(V^{(B)})^*$ do not satisfy the autonomy requirement. Arbitrary sets belonging to V^B are functions that take states to *sets* of arbitrary sets.[19] So here, too, one may have reservations about whether the autonomy requirement is met.

The logical and conceptual autonomy requirement is hard to meet. It is not even clear whether category theory meets it: some argue that the concept of a *mapping* is ultimately parasitic on the concept of set. At any rate, most multiverse theories do not meet it. Multiverse theories such as Hamkins (2012) are described using standard set theory. Nonetheless, such theories are not devoid of philosophical interest; it just means that more philosophical work remains to be done.

On the proposed view, $\mathcal{V}$ is the 'ultimate' set-theoretic universe. In this sense, an *absolutist* interpretation of set theory is proposed. Nevertheless, there is an obvious connection with multiverse views such as those of Hamkins (2012), Steel (2014) and Väänänen (2014). We have seen how every state that $\mathcal{V}$ can be in determines a Boolean-valued set-theoretic universe $V^{(B)}$. Moreover, if we take an anti-chain $\mathcal{A}$ in B, then every $a \in \mathcal{A}$ determines a set-theoretic universe. The universes determined by elements of $\mathcal{A}$ will in general not be classical two-valued universes: they are Boolean-valued universes. Moreover, such universes *themselves* typically contain other universes. In this sense, the ultimate set-theoretic universe contains many 'multiverses'. So the position under consideration can be labelled *kaleidoscopic* absolutism.[20]

As a foundational mathematical theory, set theory must be sufficiently rich to carry out all of accepted mathematics, albeit sometimes in an exceedingly cumbersome way. Thus, in a naturalistic spirit, I take it as a *conditio sine qua non* that the set-theoretic universe makes *ZFC* true, and we have seen that $\mathcal{V}$ does this.

As mentioned above, there is a two-valued universe V that is canonically embedded in every $V^{(B)}$. But the idea is that our mathematical experience suggests that the set-theoretic world is not such a two-valued structure:

> [The] abundance of set-theoretic possibilities poses a serious difficulty for the universe view, for if one holds that there is a single

[19] Not everyone agrees that this is how arbitrary sets should be seen. Sam Roberts has suggested to me that arbitrary sets should rather be regarded as having different *elements* (which are again arbitrary sets) in different states. This is clearly an important thought, which needs to be explored further. But even on this interpretation, the official definition of V^B is carried out in ordinary set theory.

[20] The 'multiverses' in $V^{(B)}$ (determined by anti-chains) can be turned into multiverses of classical, two-valued universes by well-known ultrafilter techniques (Bell 2005, ch. 4).

> absolute background concept of set, then one must explain or explain away as imaginary all of the alternative universes that set theorists seem to have constructed. This seems a difficult task, for we have a robust experience in those worlds, and they appear fully set-theoretic to us. (Hamkins 2012, p. 418)

Hamkins takes the independence phenomena to be evidence for his multiverse view; I take them to be evidence for the kaleidoscopic absolutist view.

One might wonder whether it is reasonable to expect V to be a *state* of (some, or even every) $V^{(B)}$. If it is, for some $V^{(B)}$, then B will have at least one *atom* a, and $V = V_a^{(B)}$. So then $V^{(B)}$, and therefore also $\mathcal{V}$, will contain at least one maximally specific state, that is, a possible world in the Carnapian sense of the word. There are, however, reasons for believing that V is *not* a state of a $V^{(B)}$ in the range of $\mathcal{V}$. If for some $a \in B$, $V = V_a^{(B)}$, then there is at least one completely classical state that the set-theoretic universe can be in. Moreover, this state is then also the *only* fully determinate state that the set-theoretic universe can be in. Set-theoretic experience provides no reason to think that there is any such super-special universe that the set-theoretic universe can be.

The general picture is then as follows. Set-theoretic experience—forcing, combined with large cardinal axioms, infinitary combinatorics, and so on—suggests that there are *many* states that the set-theoretic universe can be in. So $\mathcal{V}$ has to be such that it can be in all and only those states. And this imposes restrictions on what V is like and what B can be like. We have seen that for any $V^{(B)}$ in the value range of $\mathcal{V}$, the Boolean algebra B should probably not be the $\{0, 1\}$-algebra.

Beyond this, matters are less clear. Since we want universes with at least some large cardinals to be possibilities, we probably do not want V to be Gödel's constructive universe L. Perhaps it can be argued that V contains many large cardinals. If indeed $V \neq \mathrm{L}$, then elementary considerations concerning forcing show that $\mathcal{V}$ cannot be in the state of being L (rather than L just being *definable* in some such state). I take this to be in agreement with the fact that the existence of large cardinals is much less controversial than, for instance, the assertion that the continuum hypothesis is true (or the assertion that it is false). The fact that we have only very limited knowledge of what the $V^{(B)}$s in the range of $\mathcal{V}$ are does not, however, preclude us from drawing some conclusions about truth-value determinateness beyond the theorems of ZFC. For instance, since forcing techniques show that CH is independent even of set theory with large cardinals, we have good reasons to believe that CH does not have a determinate truth-value.

6. Identity

In Boolean-valued models we can have $[\![\xi = \eta]\!] = a$ for some Boolean value $0 < a < 1$. So it seems that we are committing ourselves to identity being to some degree an *indeterminate* relation.

Evans (1978) held that indeterminacy of identity is incoherent. His argument is a simple *reductio* based on Leibniz's principle of the indiscernibility of identicals. Consider any ξ and η that are not determinately identical. Then ξ has a property, namely, being identical to ξ, that η does not have. So ξ and η are determinately different from each other.

It has been observed that, strictly speaking, Evans's argument does not go through. In Evans's argument, Leibniz's principle is applied to the predicate $\lambda z[z$ is identical with $\xi]$. But then we can only conclude that ξ and η are not identical, not that they are *determinately* non-identical.

However, Williamson (2005, §8) has shown how an Evans-like argument can nonetheless be carried through with the use of two further plausible principles. First, the following inference rule seems valid:

> From a proof of $\phi \rightarrow \psi$, infer that if it is determinately the case that ϕ, then it is determinately the case that ψ.

Moreover, if it is determinately the case that ϕ, then ψ. Using these proof principles, Evans's argument can validly be strengthened to conclude that there can be no ξ, η, such that (i) it is determinately the case that they are not determinately identical, and (ii) it is also determinately the case that they are not determinately different.

The moral that is often taken from arguments such as these is that there is no ontological vagueness but only semantic and epistemic vagueness. That is, I surmise, also the attitude that set theorists habitually take, and it is one of the main reasons why the $V^{(B)}$s other than V are not taken to be candidates for being the 'real' mathematical universe. Indeed, in the forcing poset approach, the vagueness involved is pretty much *officially* regarded as semantic in nature, for the counterparts of the 'ontologically vague' sets in the Boolean-valued approach are the $\mathbb{P}$-*names* (Kunen 1980, ch. 7, §2).

I believe that the received view, that there is only semantic and epistemic vagueness, is correct. I will now argue that this view is compatible with the foundational proposal explored in the present article.

Clearly there are many 'correlated' pairs ξ, η of Boolean-valued sets that are not numerically identical to each other but that are 'judged' to be identical by certain Boolean-valued models. Here is a simple example.

Example 14. Consider the simple Boolean algebra $B_0 = \{0, a, b, 1\}$ with $0 < a, b < 1$ and $a \bot b$. Let $\xi = \{\varnothing \to 1, (\varnothing \to 1) \to 1\}$, meaning that $\mathrm{Dom}(\xi) = \{\varnothing, \varnothing \to 1\}$ and $\xi(\varnothing) = \xi(\varnothing \to 1) = 1$. Moreover, let $\eta = \{u \to 1, v \to 1\}$, with u and v being the following 'anti-correlated' sets:

- $u = \{\varnothing \to a, (\varnothing \to 1) \to b\}$
- $v = \{\varnothing \to b, (\varnothing \to 1) \to a\}$

Then clearly we have, in the strict sense, $\xi \neq \eta$. Nonetheless, a routine but tedious calculation shows that $V^{(B_0)} \vDash \xi = \eta$.

In a Boolean-valued class model, the identity symbol '=' expresses a congruence relation other than the *real* identity relation: it 'measures' the states in which its arguments *coincide*. On the proposal under consideration, some Boolean-valued class models are good interpretations of the language of set theory. Therefore the proposed view is committed to the claim that in set theory, the symbol '=' does not express the real identity relation. As a consequence, it is not threatened by Evans's argument, nor by Williamson's modification of it. It is at the metaphysical level, that is, in arbitrary object theory, that we truthfully say that $\xi \neq \eta$; in set theory, we truthfully say that $\xi = \eta$. So in these two contexts we do not use the identity symbol with the same meaning.

From the debate about mathematical structuralism, we are familiar with the claim that in many areas of mathematics the identity symbol is sometimes not used to express the metaphysical relation of identity—remember the slogan 'identity is isomorphism'. Set theory, as a foundational discipline, is often taken to be an exception to this phenomenon. The reason is that the isomorphism-as-identity phenomenon is taken to appear only when identifications *across structures* are made, for instance, when a substructure of the real numbers is identified with the natural numbers. On the view explored in the present article, this is not correct. Set theory is only concerned with one single structure. Nonetheless, according to the arbitrary object interpretation of set theory, even in the official language of set theory, the identity symbol does not express the real identity relation.

7. In closing

On the view that I have sketched, *there is a set-theoretic universe*. Moreover, as an arbitrary object, *it is an abstract entity*. It seems natural to say that on the proposed conception, the set-theoretic universe is *mind-independent*. The combination of these three commitments makes the position under consideration a form of *mathematical platonism*. At the same time, this position rejects a strong form of *truth-value realism* according to which every set-theoretic statement has exactly one of the traditional truth-values (true, false). However, it is now fairly generally recognized that mathematical platonism per se is not committed to this extra thesis, even though most traditional forms of mathematical platonism do sign up to it.

Versions of set-theoretical platonism without truth-value realism have been proposed in the literature. In this article, I have suggested one particular such view that takes the set-theoretic universe and the sets in it to be arbitrary objects. I do not claim that the view that I have proposed is the *only* way in which forcing models can metaphysically be related to arbitrary object theory. I will close by outlining the contours of what may be an alternative way of seeing elements of forcing models as arbitrary objects.

The construction of a forcing model is sometimes seen as analogous to the process of adjoining an object to an algebraic structure (see, for instance, Chow 2008, p. 2). For definiteness, consider the construction of the ring of polynomials in one variable over $\mathbb{R}$. In terms of arbitrary object theory, this process can be seen roughly as follows. We start with an arbitrary object X with value range $\mathbb{R}$. Then we consider all arbitrary objects that *depend* on X, in the sense of being polynomially determined by X. The resulting collection of arbitrary objects forms a ring.

Similarly, given a poset $\mathbb{P}$ in a countable model set theory, a generic filter G can *in a sense* be taken to be an arbitrary subset of $\mathbb{P}$. This is how Fine himself appears to interpret the role of arbitrary objects in forcing:

> It sometimes appears as if a mathematician is making significant use of an arbitrary or 'generic' object. Obvious examples are the use of generic sets in the independence proofs … (Fine 1983, p. 74)

More specifically, the filter G can be taken to be arbitrary in the sense that it intersects every dense subset of $\mathbb{P}$, where dense subsets of $\mathbb{P}$ are taken to express 'typical' properties (Venturi 2019). Given such an 'arbitrary' subset G of $\mathbb{P}$, the forcing extension $M[G]$ can then be regarded as a collection of *dependent* arbitrary objects (for they depend on G).

This way of connecting forcing models with arbitrary object theory makes use of a distinction between *dependent* and *independent* arbitrary objects. Such a distinction does not figure in the theory of arbitrary objects used in the present article.[21] But it is the cornerstone of Fine's arbitrary object theory. So this alternative account is perhaps best developed fully within the framework of Fine's theory.

Observe, however, that the sense of arbitrariness that is operative in this alternative account is somewhat different from the one explored in the present article. We have seen how in the Boolean-valued model approach, the elements of the complete Boolean algebra play the role of the states. Given the nature of the mathematical correspondence between the Boolean-valued model approach and the poset approach, it must be the elements of the poset $\mathbb{P}$ that play the role of the states in the latter approach. Indeed, when $q \leq p$ in a poset $\mathbb{P}$, this is commonly described as 'q being a refinement of p'. It is therefore not the filter G, but the standard *name* of G in M that must be seen as an arbitrary object in M according to the approach developed in the present article.[22]

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[21] Not fundamentally, anyway; but notions of dependence can be *defined* in the framework that was used in the present paper: see Horsten (2019, §9.4).

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